BEREZIN TRANSFORMS ON NONCOMMUTATIVE POLYDOMAINS

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Abstract. This paper is an attempt to unify the multivariable operator model theory for ball-like domains and commutative polydiscs, and extend it to a more general class of noncommutative polydomains $\mathbf{D}_{\mathbf{q}}^{\mathbf{m}}(\mathcal{H})$ in $B(\mathcal{H})^n$. An important role in our study is played by noncommutative Berezin transforms associated with the elements of the polydomain. These transforms are used to prove that each such polydomain has a universal model $\mathbf{W} = \{\mathbf{W}_{i,j}\}$ consisting of weighted shifts acting on a tensor product of full Fock spaces. We introduce the noncommutative Hardy algebra $F^\infty(\mathbf{D}_{\mathbf{q}}^{\mathbf{m}})$ as the weakly closed algebra generated by $\{\mathbf{W}_{i,j}\}$ and the identity, and use it to provide a WOT-continuous functional calculus for completely non-coisometric tuples in $\mathbf{D}_{\mathbf{q}}^{\mathbf{m}}(\mathcal{H})$, which are identified. It is shown that the Berezin transform is a completely isometric isomorphism between $F^\infty(\mathbf{D}_{\mathbf{q}}^{\mathbf{m}})$ and the algebra of bounded free holomorphic functions on the radial part of $\mathbf{D}_{\mathbf{q}}^{\mathbf{m}}(\mathcal{H})$. A characterization of the Beurling type joint invariant subspaces under $\{\mathbf{W}_{i,j}\}$ is also provided.

It has been an open problem for quite some time to find significant classes of elements in the commutative polidisc for which a theory of characteristic functions and model theory can be developed along the lines of the Sz.-Nagy–Foias theory of contractions. We give a positive answer to this question, in our more general setting, providing a characterization for the class of tuples of operators in $\mathbf{D}_{\mathbf{q}}^{\mathbf{m}}(\mathcal{H})$ which admit characteristic functions. The characteristic function is constructed explicitly as an artifact of the noncommutative Berezin kernel associated with the polydomain, and it is proved to be a complete unitary invariant for the class of completely non-coisometric tuples. Using noncommutative Berezin transforms and C^*-algebras techniques, we develop a dilation theory on the noncommutative polydomain $\mathbf{D}_{\mathbf{q}}^{\mathbf{m}}(\mathcal{H})$.

Contents

Introduction

Throughout this paper, we denote by $B(\mathcal{H})$ the algebra of bounded linear operators on a Hilbert space $\mathcal{H}$. A polynomial $q \in \mathbb{C}[Z_1, \ldots, Z_n]$ in n noncommuting indeterminates is called positive regular if all its coefficients are positive, the constant term is zero, and the coefficients of the linear terms $Z_1, \ldots, Z_n$ are different from zero. If $X = (X_1, \ldots, X_n) \in B(\mathcal{H})^n$ and $q = \sum_\alpha a_\alpha Z_\alpha$, we define the map $\Phi_{q,X} : B(\mathcal{H}) \to B(\mathcal{H})$ by setting $\Phi_{q,X}(Y) := \sum_\alpha a_\alpha X_\alpha Y X_\alpha^*$.

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Given two k-tuples $\mathbf{m} := (m_1, \ldots, m_k)$ and $\mathbf{n} := (n_1, \ldots, n_k)$ with $m_i, n_i \in \mathbb{N} := \{1, 2, \ldots\}$, and a k-tuple $\mathbf{q} = (q_1, \ldots, q_k)$ of positive regular polynomials $q_i \in \mathbb{C}[Z_1, \ldots, Z_{n_i}]$, we associate with each element $\mathbf{X} = (X_1, \ldots, X_k) \in B(\mathcal{H})^{n_1} \times \cdots \times B(\mathcal{H})^{n_k}$ the *defect mapping* $\mathbf{\Delta}^{\mathbf{m}}_{\mathbf{q},\mathbf{X}} : B(\mathcal{H}) \to B(\mathcal{H})$ defined by

$$\mathbf{\Delta}^{\mathbf{m}}_{\mathbf{q},\mathbf{X}} := (id - \Phi_{q_1,X_1})^{m_1} \circ \cdots \circ (id - \Phi_{q_k,X_k})^{m_k}.$$

We denote by $B(\mathcal{H})^{n_1} \times_c \cdots \times_c B(\mathcal{H})^{n_k}$ the set of all tuples $\mathbf{X} = (X_1, \ldots, X_k) \in B(\mathcal{H})^{n_1} \times \cdots \times B(\mathcal{H})^{n_k}$, where $X_i := (X_{i,1}, \ldots, X_{i,n_i}) \in B(\mathcal{H})^{n_i}$, $i \in \{1, \ldots, k\}$, with the property that, for any $p, q \in \{1, \ldots, k\}$, $p \neq q$, the entries of X_p are commuting with the entries of X_q. In this case we say that X_p and X_q are commuting tuples of operators. Note that the operators $X_{i,1}, \ldots, X_{i,n_i}$ are not necessarily commuting.

In this paper, we develop an operator model theory and a theory of free holomorphic functions on the noncommutative polydomains

$$\mathbf{D}^{\mathbf{m}}_{\mathbf{q}}(\mathcal{H}) := \left\{ \mathbf{X} = (X_1, \ldots, X_k) \in B(\mathcal{H})^{n_1} \times_c \cdots \times_c B(\mathcal{H})^{n_k} : \ \mathbf{\Delta}^{\mathbf{p}}_{\mathbf{q},\mathbf{X}}(I) \geq 0 \ \text{ for } \ \mathbf{0} \leq \mathbf{p} \leq \mathbf{m} \right\}.$$

Our study is an attempt to unify the multivariable operator model theory for the ball-like domains and commutative polydiscs, and to extend it further to the above-mentioned polydomains. The main tool in our investigation is a Berezin [13] type transform associated with the *abstract noncommutative domain* $\mathbf{D}^{\mathbf{m}}_{\mathbf{q}} := \{\mathbf{D}^{\mathbf{m}}_{\mathbf{q}}(\mathcal{H}) : \ \mathcal{H} \text{ is a Hilbert space}\}$.

In the last sixty years, this type of polydomains has been studied in several particular cases. Most of all, we mention the study of the closed operator unit ball

$$[B(\mathcal{H})]^-_1 := \{X \in B(\mathcal{H}) : \ I - XX^* \geq 0\}$$

(which corresponds to the case $k = n_1 = m_1 = 1$, and $q_1 = Z$) which has generated the celebrated Sz.-Nagy–Foias [54] theory of contractions on Hilbert spaces and has had profound implications in function theory, interpolation, and linear systems theory. When $k = n_1 = 1$, $m_1 \geq 2$, and $q_1 = Z$, the corresponding domain coincides with the set of all m-hypercontractions studied by Agler in [1], [2], and recently by Olofsson [29], [30].

In several variables, the case when $k = 1$, $n_1 \geq 2$, $m_1 = 1$, and $q_1 = Z_1 + \cdots + Z_n$, corresponds to the closed operator ball

$$[B(\mathcal{H})^n]^-_1 := \{(X_1, \ldots, X_n) \in B(\mathcal{H})^n : \ I - X_1X_1^* - \cdots - X_nX_n^* \geq 0\}$$

and its study has generated a *free* analogue of Sz.-Nagy–Foias theory (see [22], [14], [33], [34], [35], [36], [37], [38], [39], [19], [20], [8], [41], [43], [47], and the references there in). The commutative case was considered by Drurry [21], extensively studied by Arveson [5], [6], and also in [38], [41], [9], and [10]. We should remark that, in recent years, many results concerning the theory of row contractions were extended by Muhly and Solel ([26], [27], [28]) to representations of tensor algebras over C^*-correspondences and Hardy algebras. We mention that in the particular case when $k = 1$ and q_1 is a positive regular polynomial, the corresponding domain was studied in [46], if $m_1 = 1$, and in [42], [48], [49], when $m_1 \geq 2$. The commutative case when $m_1 \geq 2$, $n_1 \geq 2$, and $q_1 = Z_1 + \cdots + Z_n$, was studied by Athavale [7], Müller [24], Müller-Vasilescu [25], Vasilescu [56], and Curto-Vasilescu [16]. Some of these results were extended by S. Pott [51] when q_1 is a positive regular polynomial in commuting indeterminates.

The commutative polydisc case, i.e, $k \geq 2$, $n_1 = \cdots = n_k = 1$, and $\mathbf{q} = (Z_1, \ldots, Z_k)$, was first considered by Brehmer [15] in connection with regular dilations. Motivated by Agler's work [2] on weighted shifts as model operators, Curto and Vasilescu developed a theory of standard operator models in the polydisc in [17], [18]. Timotin [55] was able to obtain some of their results from Brehmer's theorem. The polyball case, when $k \geq 2$ and $q_i = Z_1 + \cdots + Z_{n_i}$, $i \in \{1, \ldots, k\}$, was considered in [38] and [11] for the noncommutative and commutative case, respectively. As far as we know, unlike the ball case, there is no theory of characteristic functions, analoguos to the Sz.-Nagy–Foias theory, for significant classes of operators in the polydisc (or polyball) case.

In Section 1, we work out some basic properties of the noncommutative polydomains $\mathbf{D}^{\mathbf{m}}_{\mathbf{q}}(\mathcal{H})$. One of the main results, which plays an important role in the present paper, states that any podydomain $\mathbf{D}^{\mathbf{m}}_{\mathbf{q}}(\mathcal{H})$ is *radial*, i.e., $r\mathbf{X} \in \mathbf{D}^{\mathbf{m}}_{\mathbf{q}}(\mathcal{H})$ whenever $\mathbf{X} \in \mathbf{D}^{\mathbf{m}}_{\mathbf{q}}(\mathcal{H})$ and $r \in [0, 1)$. This fact has also an important consequence in the particular case when $k = 1$, namely, that all the results from [42], [48], [49], which were proved in the setting of the radial part of $\mathbf{D}^{m_1}_{q_1}(\mathcal{H})$, are true for any domain $\mathbf{D}^{m_1}_{q_1}(\mathcal{H})$.

In Section 2, we introduce the *noncommutative Berezin transform* at $\mathbf{T} \in \mathbf{D}_{\mathbf{q}}^{\mathbf{m}}(\mathcal{H})$ to be the mapping $\mathbf{B}_{\mathbf{T}} : B(\otimes_{i=1}^k F^2(H_{n_i})) \to B(\mathcal{H})$ defined by

$$\mathbf{B}_{\mathbf{T}}[g] := \mathbf{K}_{\mathbf{q},\mathbf{T}}^*(g \otimes I_{\mathcal{H}})\mathbf{K}_{\mathbf{q},\mathbf{T}}, \qquad g \in B(\otimes_{i=1}^k F^2(H_{n_i})),$$

where $F^2(H_{n_i})$ is the full Fock space on n_i generators and

$$\mathbf{K}_{\mathbf{q},\mathbf{T}} : \mathcal{H} \to F^2(H_{n_1}) \otimes \cdots \otimes F^2(H_{n_k}) \otimes \overline{\mathbf{\Delta}_{\mathbf{q},\mathbf{T}}^{\mathbf{m}}(I)(\mathcal{H})}$$

is the *noncommutative Berezin kernel* associated with $\mathbf{T}$, which is defined in terms of the coefficients of the positive regular polynomials $q_1, \ldots, q_k$. We remark that in the particular case when $\mathcal{H} = \mathbb{C}$, $\mathbf{q} = (Z_1, \ldots, Z_k)$, $\mathbf{T} = \lambda = (\lambda_1, \ldots, \lambda_k) \in \mathbb{D}^k$, and $m_i = n_i = 1$ for any $i \in \{1, \ldots, k\}$, we recover the Berezin transform of a bounded linear operator on the Hardy space $H^2(\mathbb{D}^k)$, i.e.,

$$\mathbf{B}_\lambda[g] = \prod_{i=1}^k (1 - |\lambda_i|^2) \langle g k_\lambda, k_\lambda \rangle, \qquad g \in B(H^2(\mathbb{D}^k)),$$

where $k_\lambda(z) := \prod_{i=1}^k (1 - \overline{\lambda}_i z_i)^{-1}$ and $z = (z_1, \ldots, z_k) \in \mathbb{D}^k$.

The noncommutative Berezin transforms are used to prove the main result of this section (Theorem 2.2) which shows that each polydomain $\mathbf{D}_{\mathbf{q}}^{\mathbf{m}}(\mathcal{H})$ has a universal model $\mathbf{W} = \{\mathbf{W}_{i,j}\}$ consisting of weighted shifts acting on a tensor product of full Fock spaces. Moreover, we show that a tuple of operators $\mathbf{X}$ is in the noncommutative polydomain $\mathbf{D}_{\mathbf{q}}^{\mathbf{m}}(\mathcal{H})$ if and only if there exists a completely positive linear map $\Psi : C^*(\mathbf{W}_{i,j}) \to B(\mathcal{H})$ such that

$$\Psi(p(\mathbf{W})r(\mathbf{W})^*) = p(X)r(X)^*,$$

for any $p(\mathbf{W}), r(\mathbf{W})$ polynomials in $\{\mathbf{W}_{i,j}\}$ and the identity.

In Section 3, we introduce the noncommutative Hardy algebra $F^\infty(\mathbf{D}_{\mathbf{q}}^{\mathbf{m}})$ as the weakly closed algebra generated by $\{\mathbf{W}_{i,j}\}$ and the identity, and use it to provide a WOT-continuous functional calculus for *completely non-coisometric* tuples $\mathbf{T} = \{T_{i,j}\}$ in $\mathbf{D}_{\mathbf{q}}^{\mathbf{m}}(\mathcal{H})$, which are identified. We show that

$$\Phi(\varphi) := \text{SOT-}\lim_{r \to 1} \varphi(rT_{i,j}), \qquad \varphi = \varphi(\mathbf{W}_{i,j}) \in F^\infty(\mathbf{D}_{\mathbf{q}}^{\mathbf{m}}),$$

exists in the strong operator topology and defines a map $\Phi : F^\infty(\mathbf{D}_{\mathbf{q}}^{\mathbf{m}}) \to B(\mathcal{H})$ with the property that $\Phi(\varphi) = \text{SOT-}\lim_{r \to 1} \mathbf{B}_{r\mathbf{T}}[\varphi]$, where $\mathbf{B}_{r\mathbf{T}}$ is the noncommutative Berezin transform at $r\mathbf{T} \in \mathbf{D}_{\mathbf{q}}^{\mathbf{m}}(\mathcal{H})$. Moreover, Φ is a unital completely contractive homomorphism, which is WOT-continuous (resp. SOT-continuous) on bounded sets.

In Section 4, we introduce the algebra $Hol(\mathbf{D}_{\mathbf{q},\text{rad}}^{\mathbf{m}})$ of all free holomorphic functions on the *abstract radial polydomain* $\mathbf{D}_{\mathbf{q},\text{rad}}^{\mathbf{m}}$. We identify the *polydomain algebra* $\mathcal{A}(\mathbf{D}_{\mathbf{q}}^{\mathbf{m}})$ (the closed algebra generated by $\{\mathbf{W}_{i,j}\}$ and the identity) and the Hardy algebra $F^\infty(\mathbf{D}_{\mathbf{q}}^{\mathbf{m}})$ with subalgebras of $Hol(\mathbf{D}_{\mathbf{q},\text{rad}}^{\mathbf{m}})$. For example, it is shown that the noncommutative Berezin transform is a completely isometric isomorphism between $F^\infty(\mathbf{D}_{\mathbf{q}}^{\mathbf{m}})$ and the algebra of bounded free holomorphic functions on $\mathbf{D}_{\mathbf{q},\text{rad}}^{\mathbf{m}}$. We remark that there is an important connection between the theory of free holomorphic functions on abstract radial polydomains $\mathbf{D}_{\mathbf{q},\text{rad}}^{\mathbf{m}}$, and the theory of holomorphic functions on polydomains in $\mathbb{C}^d$ (see [23], [52]). Indeed, if $\mathcal{H} = \mathbb{C}^p$ and $p \in \mathbb{N}$, then $\mathbf{D}_{\mathbf{q}}^{\mathbf{m}}(\mathbb{C}^p)$ can be seen as a subset of $\mathbb{C}^{(n_1+\cdots+n_k)p^2}$ with an arbitrary norm. Given a free holomorphic function φ on the abstract radial polydomain $\mathbf{D}_{\mathbf{q},\text{rad}}^{\mathbf{m}}$, we prove that its representation on $\mathbb{C}^p$, i.e., the map $\widehat{\varphi}$ defined by

$$\mathbb{C}^{(n_1+\cdots+n_k)p^2} \supset \mathbf{D}_{\mathbf{q},\text{rad}}^{\mathbf{m}}(\mathbb{C}^p) \ni (\lambda_{i,j}) \mapsto \varphi(\lambda_{i,j}) \in \mathbb{C}^{p^2}$$

is a holomorphic function on the interior of $\mathbf{D}_{\mathbf{q}}^{\mathbf{m}}(\mathbb{C}^p)$. In addition, $\widehat{\varphi}$ is bounded when $\varphi \in F^\infty(\mathbf{D}_{\mathbf{q}}^{\mathbf{m}})$, and it has continuous extension to $\mathbf{D}_{\mathbf{q}}^{\mathbf{m}}(\mathbb{C}^p))$ when $\varphi \in \mathcal{A}(\mathbf{D}_{\mathbf{q}}^{\mathbf{m}})$.

In Section 5, we obtain a characterization of the Beurling [12] type joint invariant subspaces under $\{\mathbf{W}_{i,j}\}$. We prove that a subspace $\mathcal{M} \subset \otimes_{i=1}^k F^2(H_{n_i}) \otimes \mathcal{H}$ has the form $\mathcal{M} = \Psi\left((\otimes_{i=1}^k F^2(H_{n_i})) \otimes \mathcal{E}\right)$ for some *inner multi-analytic operator* with respect to the universal model $\mathbf{W}$, if and only if

$$\mathbf{\Delta}_{\mathbf{q},\mathbf{W}\otimes\mathbf{I}}^{\mathbf{p}}(P_{\mathcal{M}}) \geq 0, \qquad \text{for any } \ \mathbf{p} \in \mathbb{Z}_+^k, \mathbf{p} \leq \mathbf{m},$$

where $P_\mathcal{M}$ is the orthogonal projection of the Hilbert space $\otimes_{i=1}^k F^2(H_{n_i})\otimes \mathcal{H}$ onto $\mathcal{M}$. In the particular case when $\mathbf{m}=(1,\dots,1)$, the latter condition is satisfied when $\mathbf{W}\otimes I|_\mathcal{M}$ is a doubly commuting tuple. We also characterize the reducing subspaces under $\{\mathbf{W}_{i,j}\}$ and present several results concerning the model theory for pure elements in the noncommutative polydomain $\mathbf{D_q^m}(\mathcal{H})$.

In Section 6, we provide a characterization for the class of tuples of operators in $\mathbf{D_q^m}(\mathcal{H})$ which admit characteristic functions. We say that $\mathbf{T}\in \mathbf{D_q^m}(\mathcal{H})$ has characteristic function if there is a multi-analytic operator Ψ with respect to the universal model $\mathbf{W}$ such that

$$\mathbf{K_{q,T}}\mathbf{K_{q,T}^*}+\Psi\Psi^*=I,$$

where $\mathbf{K_{q,T}}$ is the noncommutative Berezin kernel associated with $\mathbf{D_q^m}(\mathcal{H})$. In this case, Ψ is essentially unique. We prove that $\mathbf{T}\in \mathbf{D_q^m}(\mathcal{H})$ has characteristic function if and only if

$$\mathbf{\Delta^p_{q,W\otimes I}}\left(I-\mathbf{K_{q,T}}\mathbf{K_{q,T}^*}\right)\geq 0,\qquad \text{for any } \ \mathbf{p}\in \mathbb{Z}_+^k, \mathbf{p}\leq \mathbf{m}.$$

The characteristic function is constructed explicitly and it is proved to be a complete unitary invariant for the class of completely non-coisometric tuples. Moreover, we provide an operator model for this class of elements in $\mathbf{D_q^m}(\mathcal{H})$ in terms of their characteristic functions.

In Section 7, using several results from the previous sections and C^*-algebras techniques, we develop a dilation theory on the noncommutative polydomain $\mathbf{D_q^m}(\mathcal{H})$. The main result states that if $\mathbf{T}=\{T_{i,j}\}$ is a tuple in $\mathbf{D_q^m}(\mathcal{H})$, then there exists a $*$-representation $\pi: C^*(\mathbf{W}_{i,j})\to B(\mathcal{K}_\pi)$ on a separable Hilbert space $\mathcal{K}_\pi$, which annihilates the compact operators and $\mathbf{\Delta^m_{q,\pi(W)}}(I_{\mathcal{K}_\pi})=0$ such that $\mathcal{H}$ can be identified with a $*$-cyclic co-invariant subspace of

$$\tilde{\mathcal{K}}:=\left[(\otimes_{i=1}^k F^2(H_{n_i}))\otimes \overline{\mathbf{\Delta^m_{q,T}}(I)(\mathcal{H})}\right]\oplus \mathcal{K}_\pi$$

under each operator

$$V_{i,j}:=\begin{bmatrix}\mathbf{W}_{i,j}\otimes I & 0\\ 0 & \pi(\mathbf{W}_{i,j})\end{bmatrix},$$

and such that $T_{i,j}^*=V_{i,j}^*|_\mathcal{H}$ for all i,j. Under a certain additional condition on the universal model $\mathbf{W}$, the dilation above is minimal and unique up to unitary equivalence. We also obtain Wold type decompositions for non-degenerate $*$-representations of the C^*-algebra $C^*(\mathbf{W}_{i,j})$.

We mention that the results of this paper are presented in a more general setting, when $\mathbf{q}$ is replaced by a k-tuple $\mathbf{f}=(f_1,\dots,f_k)$ of positive regular free holomorphic functions in a neighborhood of the origin. Also, the results are used in [50] to develop an operator model theory for varieties in noncommutative polydomains. This includes various commutative cases which are presented in close connection with the theory of holomorphic functions in several complex variables.

1. A class of noncommutative polydomains

For each $i\in\{1,\dots,k\}$, let $\mathbb{F}_{n_i}^+$ be the unital free semigroup on n_i generators $g_1^i,\dots,g_{n_i}^i$ and the identity g_0^i. The length of $\alpha\in\mathbb{F}_{n_i}^+$ is defined by $|\alpha|:=0$ if $\alpha=g_0^i$ and $|\alpha|:=p$ if $\alpha=g_{j_1}^i\cdots g_{j_p}^i$, where $j_1,\dots,j_p\in\{1,\dots,n_i\}$. If $Z_1,\dots,Z_{n_i}$ are noncommuting indeterminates, we denote $Z_\alpha:=Z_{j_1}\cdots Z_{j_p}$ and $Z_{g_0^i}:=1$. Let $f_i:=\sum_{\alpha\in\mathbb{F}_{n_i}^+} a_{i,\alpha}Z_\alpha$, $\ a_{i,\alpha}\in\mathbb{C}$, be a formal power series in n_i noncommuting indeterminates $Z_1,\dots,Z_{n_i}$. We say that f_i is a *positive regular free holomorphic function* if the following conditions hold: $a_{i,\alpha}\geq 0$ for any $\alpha\in\mathbb{F}_{n_i}^+$, $\ a_{i,g_0^i}=0$, $a_{i,g_j^i}>0$ for $j=1,\dots,n_i$, and

$$\limsup_{k\to\infty}\left(\sum_{|\alpha|=k}|a_{i,\alpha}|^2\right)^{1/2k}<\infty.$$

Given $X_i:=(X_{i,1},\dots,X_{i,n_i})\in B(\mathcal{H})^{n_i}$, define the map $\Phi_{f_i,X_i}:B(\mathcal{H})\to B(\mathcal{H})$ by setting

$$\Phi_{f_i,X_i}(Y):=\sum_{k=1}^\infty\sum_{\alpha\in\mathbb{F}_{n_i}^+,|\alpha|=k} a_{i,\alpha}X_{i,\alpha}YX_{i,\alpha}^*,\qquad Y\in B(\mathcal{H}),$$

where the convergence is in the week operator topology.

Let $\mathbf{n} := (n_1, \ldots, n_k)$ and $\mathbf{m} := (m_1, \ldots, m_k)$, where $n_i, m_i \in \mathbb{N} := \{1, 2, \ldots\}$ and $i \in \{1, \ldots, k\}$, and let $\mathbf{f} := (f_1, \ldots, f_k)$ be a k-tuple of positive regular free holomorphic functions. We introduce the noncommutative polydomain $\mathbf{D}_{\mathbf{f}}^{\mathbf{m}}(\mathcal{H})$ to be the set of all k-tuples

$$\mathbf{X} := (X_1, \ldots, X_k) \in B(\mathcal{H})^{n_1} \times_c \cdots \times_c B(\mathcal{H})^{n_k}$$

with the property that $\Phi_{f_i,X_i}(I) \le I$ and

$$(id - \Phi_{f_1,X_1})^{\epsilon_1 m_1} \cdots (id - \Phi_{f_k,X_k})^{\epsilon_k m_k}(I) \ge 0$$

for any $i \in \{1, \ldots, k\}$ and $\epsilon_i \in \{0, 1\}$. We use the convention that $(id - \Phi_{f_i,X_i})^0 = id$. We remark that $\mathbf{D}_{\mathbf{f}}^{\mathbf{m}}(\mathcal{H})$ contains a polyball $[B(\mathcal{H})^{n_1}]_{r_1}^- \times_c \cdots \times_c [B(\mathcal{H})^{n_k}]_{r_k}^-$ for some $r_1, \ldots, r_k > 0$, where

$$[B(\mathcal{H})^{n_i}]_{r_i}^- := \{(Y_1, \ldots, Y_{n_i}) \in B(\mathcal{H})^{n_i} : \ Y_1 Y_1^* + \cdots + Y_{n_i} Y_{n_i}^* \le r_i^2 I\}.$$

Throughout this paper, we refer to $\mathbf{D}_{\mathbf{q}}^{\mathbf{m}} := \{\mathbf{D}_{\mathbf{q}}^{\mathbf{m}}(\mathcal{H}) : \ \mathcal{H}\text{is a Hilbert space}\}$ as the abstract noncommutative polydomain, and $\mathbf{D}_{\mathbf{q}}^{\mathbf{m}}(\mathcal{H})$ as its representation on the Hilbert space $\mathcal{H}$.

A linear map $\varphi : B(\mathcal{H}) \to B(\mathcal{H})$ is called power bounded if there exists a constant $M > 0$ such that $\|\varphi^k\| \le M$ for any $k \in \mathbb{N}$. For information on completely bounded (resp. positive) maps, we refer to [31] and [32]. If $\mathbf{p} := (p_1, \ldots, p_k) \in \mathbb{Z}_+^k$ and $\mathbf{q} := (q_1, \ldots, q_k) \in \mathbb{Z}_+^k$, we set $\mathbf{p} \le \mathbf{q}$ iff $p_i \le q_i$ for all $i \in \{1, \ldots, k\}$, where $\mathbb{Z}_+ := \{0, 1, \ldots\}$.

Proposition 1.1. *Let $\varphi_i : B(\mathcal{H}) \to B(\mathcal{H})$, $i \in \{1, \ldots, k\}$, be power bounded positive linear maps such that*

$$\varphi_i \varphi_j = \varphi_j \varphi_i, \qquad i, j \in \{1, \ldots, k\}.$$

If $Y \in B(\mathcal{H})$ is a self-adjoint operator and $\mathbf{p} := (p_1, \ldots, p_k) \in \mathbb{Z}_+^k$ with $p_i \ge 1$, then the following statements are equivalent.

(i) $(id-\varphi_1)^{\epsilon_1 p_1} \cdots (id-\varphi_k)^{\epsilon_k p_k}(Y) \ge 0$ *for all* $\epsilon_i \in \{0, 1\}$ *with* $\epsilon := (\epsilon_1, \ldots, \epsilon_k) \ne 0$ *and* $i \in \{1, \ldots, k\}$.
(ii) $(id - \varphi_1)^{q_1} \cdots (id - \varphi_k)^{q_k}(Y) \ge 0$ *for all* $\mathbf{q} := (q_1, \ldots, q_k) \in \mathbb{Z}_+^k$ *with* $\mathbf{q} \le \mathbf{p}$ *and* $\mathbf{q} \ne 0$.

Proof. Note that it is enough to prove that $(id - \varphi_1)^{p_1} \cdots (id - \varphi_k)^{p_k}(Y) \ge 0$ if and only if $(id - \varphi_1)^{q_1} \cdots (id - \varphi_k)^{q_k}(Y) \ge 0$ for all $\mathbf{q} := (q_1, \ldots, q_k) \in \mathbb{Z}_+^k$ with $q_i \le p_i$ and $q_i \ge 1$. We proceed by induction over $k \in \mathbb{N}$. Let $k = 1$, and assume that $(id - \varphi_1)^{p_1}(Y) \ge 0$ and $p_1 \ge 2$. Suppose that there is $h_0 \in \mathcal{H}$ such that $\left\langle (id - \varphi_1)^{p_1 - 1}(Y) h_0, h_0 \right\rangle < 0$. Set $y_j := \left\langle \varphi_1^j (id - \varphi_1)^{p_1-1}(Y) h_0, h_0 \right\rangle$, $j = 0, 1, \ldots$, and note that $\{y_j\}_{j=0}^\infty$ is a decreasing sequence with $y_j \le y_0 < 0$. Consequently, we deduce that $\sum_{j=0}^\infty y_j = -\infty$. On the other hand, we have

$$\begin{aligned} \left| \sum_{j=0}^p y_j \right| &:= \left| \left\langle (id - \varphi_1^{p+1})(id - \varphi_1)^{p_1-2}(Y) h_0, h_0 \right\rangle \right| \\ &\le \left(1 + \|\varphi_1^{p+1}(I)\| \right) \|(id - \varphi_1)^{p_1-2}(Y)\| \|h_0\|. \end{aligned}$$

Since φ_1 is power bounded, we get a contradiction. Therefore, we must have $(id - \varphi_1)^{p_1-1}(Y) \ge 0$. Continuing this process, we show that $(id-\varphi)^{p_1}(Y) \ge 0$ if and only if $(id-\varphi)^s(Y) \ge 0$ for $s = 1, 2, \ldots, p_1$. Now, assume that

$$(id - \varphi_1)^{p_1} \cdots (id - \varphi_k)^{p_k} (id - \varphi_{k+1})^{p_{k+1}}(Y) \ge 0.$$

Due to the fact that $\varphi_i \varphi_j = \varphi_j \varphi_i$ for all $i, j \in \{1, \ldots, k\}$, we deduce that $(id - \varphi_{k+1})^{p_{k+1}}(Y_k) \ge 0$, where $Y_k := (id - \varphi_1)^{p_1} \cdots (id - \varphi_k)^{p_k}(Y)$. On the other hand, due to the identity

$$(id - \varphi_k)^{p_k}(Y) = \sum_{p=0}^{p_k} (-1)^p \binom{p_k}{p} \varphi_k^p(Y),$$

the operator $(id - \varphi_k)^{p_k}(Y)$ is self-adjoint whenever φ_k is a positive linear map and Y is a self-adjoint operator. Inductively, one can easily see that Y_k is a self-adjoint operator. Now, applying the case $k = 1$,

we deduce that $(id-\varphi_{k+1})^{p_{k+1}}(Y_k)\geq 0$ if and only if $(id-\varphi_{k+1})^{q_{k+1}}(Y_k)\geq 0$ for all $q_{k+1}\in\{0,1,\ldots,p_{k+1}\}$. Hence,

$$(id-\varphi_1)^{p_1}\cdots(id-\varphi_k)^{p_k}(id-\varphi_{k+1})^{q_{k+1}}(Y)\geq 0.$$

Due to the induction hypothesis, we deduce that

$$(id-\varphi_1)^{q_1}\cdots(id-\varphi_k)^{q_k}(id-\varphi_{k+1})^{q_{k+1}}(Y)\geq 0$$

for all $(q_1,\ldots,q_{k+1})\in\mathbb{Z}_+^{k+1}$ with $q_i\leq p_i$ and $q_i\geq 1$. This completes the proof. $\square$

Let $\Phi=(\varphi_1,\ldots,\varphi_k)$ be a k-tuple of power bounded, positive linear maps on $B(\mathcal{H})$ such that $\varphi_i\varphi_j=\varphi_j\varphi_i$, $i,j\in\{1,\ldots,k\}$. For each $\mathbf{p}:=(p_1,\ldots,p_k)\in\mathbb{Z}_+^k$, we define the linear map $\Delta_\Phi^{\mathbf{p}}:B(\mathcal{H})\to B(\mathcal{H})$ by setting

$$\Delta_\Phi^{p_1,\ldots,p_k}=\Delta_\Phi^{\mathbf{p}}:=(id-\varphi_1)^{p_1}\cdots(id-\varphi_k)^{p_k}.$$

Lemma 1.2. *Let $\mathbf{m}\in\mathbb{N}^k$ and let $Y\in B(\mathcal{H})$ be a self-adjoint operator such that $\Delta_\Phi^{\mathbf{p}}(Y)\geq 0$ for all $\mathbf{p}\in\mathbb{Z}_+^k$ with $\mathbf{p}\leq\mathbf{m}$ and $\mathbf{p}\neq 0$. If $\mathbf{q}\in\mathbb{Z}_+^k$ with $\mathbf{q}\neq 0$ and $\mathbf{q}\leq\mathbf{m}$, then*

$$\Delta_\Phi^{\mathbf{m}}(Y)\leq\Delta_\Phi^{\mathbf{q}}(Y).$$

Proof. Set $\mathbf{m}:=(m_1,\ldots,m_k)\in\mathbb{N}^k$ and $\mathbf{m}':=(m_1-1,m_2,\ldots,m_k)$. Since $\Delta_\Phi^{\mathbf{m}'}(Y)\geq 0$ and φ_1 is a positive map, we deduce that

$$\Delta_\Phi^{\mathbf{m}}(Y)=\Delta_\Phi^{\mathbf{m}'}(Y)-\varphi_1(\Delta_\Phi^{\mathbf{m}'}(Y))\leq\Delta_\Phi^{\mathbf{m}'}(Y)$$

Using the fact that $\varphi_i\varphi_j=\varphi_j\varphi_i$ for $i,j\in\{1,\ldots,k\}$, one can continue this process and complete the proof. $\square$

Proposition 1.3. *Let $Y\in B(\mathcal{H})$ be a self-adjoint operator, $\mathbf{m}\in\mathbb{Z}_+^k$, $\mathbf{m}\neq 0$, and let $\Phi=(\varphi_1,\ldots,\varphi_k)$ be a k-tuple of commuting, power bounded, positive linear maps on $B(\mathcal{H})$ such that*

(i) *$\Delta_\Phi^{\mathbf{m}}(Y)\geq 0$, and*
(ii) *each φ_i is pure, i.e., $\varphi_i^p(I)\to 0$ strongly as $p\to\infty$.*

Then $\Delta_\Phi^{\mathbf{q}}(Y)\geq 0$ for any $\mathbf{q}\in\mathbb{Z}_+^k$ with $\mathbf{q}\leq\mathbf{m}$. In particular, $Y\geq 0$.

Proof. Set $\mathbf{m}':=(m_1-1,m_2,\ldots,m_k)$ and note that due to the fact that $\Delta_\Phi^{\mathbf{m}}(Y)\geq 0$ and φ_1 is a positive linear map, we have

$$0\leq\Delta_\Phi^{\mathbf{m}}(Y)=\Delta_\Phi^{\mathbf{m}'}(Y)-\varphi_1(\Delta_\Phi^{\mathbf{m}'}(Y)).$$

Hence, we deduce that $\varphi_1^p(\Delta_\Phi^{\mathbf{m}'}(Y))\leq\Delta_\Phi^{\mathbf{m}'}(Y)$ for any $p\in\mathbb{N}$. Since $\Delta_\Phi^{\mathbf{m}'}(Y)$ is a self-adjoint operator, we have

$$-\|\Delta_\Phi^{\mathbf{m}'}(Y)\|\varphi_1^p(I)\leq\varphi_1^p(\Delta_\Phi^{\mathbf{m}'}(Y))\leq\|\Delta_\Phi^{\mathbf{m}'}(Y)\|\varphi_1^p(I).$$

Now, taking into account that $\varphi_i^p(I)\to 0$ strongly as $p\to\infty$, we conclude that $\Delta_\Phi^{\mathbf{m}'}(Y)\geq 0$. Using the commutativity of $\varphi_1,\ldots,\varphi_k$, one can continue this process and complete the proof. $\square$

For each $i\in\{1,\ldots,k\}$, let $f_i:=\sum_{\alpha_i\in\mathbb{F}_{n_i}^+,|\alpha|\geq 1}a_{i,\alpha}Z_\alpha$ be a positive regular free holomorphic function in n_i variables and let $A:=(A_1,\ldots,A_n)\in B(\mathcal{H})^{n_i}$ be an n_i-tuple of operators such that $\sum_{|\alpha|\geq 1}a_{i,\alpha}A_\alpha A_\alpha^*$ is convergent in the weak operator topology. One can easily prove that the map $\Phi_{f_i,A}:B(\mathcal{H})\to B(\mathcal{H})$, defined by

$$\Phi_{f_i,A}(X)=\sum_{|\alpha|\geq 1}a_{i,\alpha}A_\alpha XA_\alpha^*,\qquad X\in B(\mathcal{H}),$$

where the convergence is in the weak operator topology, is a completely positive linear map which is WOT-continuous on bounded sets. Moreover, if $0<r<1$, then

$$\Phi_{f_i,A}(X)=\text{WOT-}\lim_{r\to 1}\Phi_{f_i,rA}(X),\qquad X\in B(\mathcal{H}).$$

These facts will be used in the proof of the next theorem.

Let $\mathbf{T} = (T_1, \ldots, T_k) \in B(\mathcal{H})^{n_1} \times_c \cdots \times_c B(\mathcal{H})^{n_k}$, where $T_i := (T_{i,1}, \ldots, T_{i,n_i}) \in B(\mathcal{H})^{n_i}$ for all $i = 1, \ldots, k$, be such that $\Phi_{f_i,T_i}(I)$ is well-defined in the weak operator topology. If $\mathbf{p} := (p_1, \ldots, p_k) \in \mathbb{Z}_+^k$ and $\mathbf{f} := (f_1, \ldots, f_k)$, we define the *defect mapping* $\mathbf{\Delta}_{\mathbf{f},\mathbf{T}}^{\mathbf{p}} : B(\mathcal{H}) \to B(\mathcal{H})$ by setting

$$\mathbf{\Delta}_{\mathbf{f},\mathbf{T}}^{\mathbf{p}} := (id - \Phi_{f_1,T_1})^{p_1} \cdots (id - \Phi_{f_k,T_k})^{p_k}.$$

Given $r \geq 0$, we set $r\mathbf{T} := (rT_1, \ldots, rT_k)$ and $rT_i := (rT_{i,1}, \ldots, rT_{i,n_i})$ for $i \in \{1, \ldots, k\}$. We say that the k-tuple $\mathbf{T}$ has the *radial property* with respect to $\mathbf{D}_{\mathbf{f}}^{\mathbf{m}}(\mathcal{H})$ if there exists $\delta \in (0, 1)$ such that $r\mathbf{T} \in \mathbf{D}_{\mathbf{f}}^{\mathbf{m}}(\mathcal{H})$ for any $r \in (\delta, 1]$.

Theorem 1.4. *Let $\mathbf{T} = (T_1, \ldots, T_k) \in B(\mathcal{H})^{n_1} \times_c \cdots \times_c B(\mathcal{H})^{n_k}$ be such that $\Phi_{f_i,T_i}(I) \leq I$ for any $i \in \{1, \ldots, k\}$, and let $\mathbf{q} \in \mathbb{Z}_+^k$ be with $\mathbf{q} \neq 0$. Then the following statements are equivalent:*

(i) $\mathbf{T} \in \mathbf{D}_{\mathbf{f}}^{\mathbf{m}}(\mathcal{H})$;

(ii) *for any $p_i \in \{0, 1, \ldots, m_i\}$ and $i \in \{1, \ldots, k\}$,*
$$(id - \Phi_{f_1,T_1})^{p_1} \cdots (id - \Phi_{f_k,T_k})^{p_k}(I) \geq 0;$$

(iii) $\mathbf{\Delta}_{\mathbf{f},r\mathbf{T}}^{\mathbf{m}}(I) \geq 0$ *for any $r \in [0, 1]$;*

(iv) *there exists $\delta \in (0, 1)$ such that $\mathbf{\Delta}_{\mathbf{f},r\mathbf{T}}^{\mathbf{m}}(I) \geq 0$ for any $r \in (\delta, 1)$;*

(v) $\mathbf{T}$ *has the radial property with respect to $\mathbf{D}_{\mathbf{f}}^{\mathbf{m}}(\mathcal{H})$.*

Proof. The equivalence of (i) with (ii) is due to Proposition 1.1, when applied to $\varphi_i = \Phi_{f_i,T_i}$. We prove that (ii) implies (iii). First, note that if $D \in B(\mathcal{H})$, $D \geq 0$, then, for each $i \in \{1, \ldots, k\}$,

$$(id - \Phi_{f_i,T_i})(D) \geq 0 \quad \Longrightarrow \quad (id - \Phi_{f_i,rT_i})(D) \geq 0, \quad r \in [0, 1]. \tag{1.1}$$

Indeed, if $\Phi_{f_i,T_i}(D) \leq D$, then $\Phi_{f_i,rT_i}(D) \leq D$ for any $r \in [0, 1]$. Now, assume that (ii) holds. If $\mathbf{p} \in \mathbb{Z}_+^k$ with $\mathbf{p} \geq e_1 := (1, 0, \ldots, 0) \in \mathbb{Z}_+^k$, then $(id - \Phi_{f_1,T_1})(\mathbf{\Delta}_{\mathbf{f},\mathbf{T}}^{\mathbf{p}-e_1}(I)) \geq 0$ for any $\mathbf{p} \in \mathbb{Z}_+^k$ with $e_1 \leq \mathbf{p} \leq \mathbf{m}$. Consequently, due to (1.1), we have

$$(id - \Phi_{f_1,rT_1})(\mathbf{\Delta}_{\mathbf{f},\mathbf{T}}^{\mathbf{p}-e_1}(I)) \geq 0 \tag{1.2}$$

for any $r \in [0, 1]$ and any $\mathbf{p} \in \mathbb{Z}_+^k$ with $e_1 \leq \mathbf{p} \leq \mathbf{m}$. Due to the commutativity of $\Phi_{f_1,T_1}, \ldots, \Phi_{f_k,T_k}$, the latter inequality is equivalent to

$$(id - \Phi_{f_1,T_1})(\mathbf{\Delta}_{\mathbf{f},\mathbf{T}}^{\mathbf{p}-2e_1}(id - \Phi_{f_1,rT_1})(I)) \geq 0$$

for any $r \in [0, 1]$ and any $\mathbf{p} \in \mathbb{Z}_+^k$ with $2e_1 \leq \mathbf{p} \leq \mathbf{m}$. Due to (1.2), we have $\mathbf{\Delta}_{\mathbf{f},\mathbf{T}}^{\mathbf{p}-2e_1}(id - \Phi_{f_1,rT_1})(I) \geq 0$ and, applying again relation (1.1), we deduce that

$$(id - \Phi_{f_1,T_1})(\mathbf{\Delta}_{\mathbf{f},\mathbf{T}}^{\mathbf{p}-3e_1}(id - \Phi_{f_1,rT_1})^2(I)) \geq 0$$

for any $r \in [0, 1]$ and any $\mathbf{p} \in \mathbb{Z}_+^k$ with $3e_1 \leq \mathbf{p} \leq \mathbf{m}$. Continuing this process, we obtain the inequality

$$(id - \Phi_{f_2,T_2})^{p_2} \cdots (id - \Phi_{f_k,T_k})^{p_k}(id - \Phi_{f_1,rT_1})^{p_1}(I) \geq 0$$

for any $\mathbf{p} \in \mathbb{Z}_+^k$ with $e_1 \leq \mathbf{p} \leq \mathbf{m}$, and any $r \in [0, 1]$. Similar arguments lead to the inequality $\mathbf{\Delta}_{\mathbf{f},r\mathbf{T}}^{\mathbf{m}}(I) \geq 0$ for any $r \in [0, 1]$. Since the implications (iii) $\Longrightarrow$ (iv) and (v) $\Longrightarrow$ (i) are clear, it remains to prove that (iv) $\Longrightarrow$ (v).

To this end, assume that there exists $\delta \in (0, 1)$ such that $\mathbf{\Delta}_{\mathbf{f},r\mathbf{T}}^{\mathbf{m}}(I) \geq 0$ for any $r \in (\delta, 1)$. Since $\Phi_{f_i,rT_i}(I) \leq rI$, it is clear that Φ_{f_i,rT_i} is pure for each $i \in \{1, \ldots, k\}$. Applying Proposition 1.3, we deduce that $\mathbf{\Delta}_{\mathbf{f},r\mathbf{T}}^{\mathbf{p}}(I) \geq 0$ for any $r \in (\delta, 1)$ and any $\mathbf{p} \in \mathbb{Z}_+^k$ with $\mathbf{p} \leq \mathbf{m}$. Note that $\mathbf{\Delta}_{\mathbf{f},r\mathbf{T}}^{\mathbf{p}}(I)$ is a linear combination of products of the form $\Phi_{f_1,rT_1}^{q_1} \cdots \Phi_{f_k,rT_k}^{q_k}(I)$, where $(q_1, \ldots, q_k) \in \mathbb{Z}_+^k$. On the other hand

$$\Phi_{f_1,T_1}^{q_1} \cdots \Phi_{f_k,T_k}^{q_k}(I) = \text{WOT-}\lim_{j \to \infty} \sum_{\substack{\alpha_i \in \mathbb{F}_{n_i}^+ \\ |\alpha_1| + \cdots + |\alpha_k| \leq j}} c_{\alpha_1,\ldots,\alpha_k} T_{1,\alpha_1} \cdots T_{k,\alpha_k} T_{k,\alpha_k}^* \cdots T_{1,\alpha_1}^* \leq I$$

for some positive constants $c_{\alpha_1,\ldots,\alpha_k} \geq 0$. Given $x \in \mathcal{H}$ and $\epsilon > 0$, there is $N_0 \in \mathbb{N}$ such that

$$\sum_{\substack{\alpha_i \in \mathbb{F}_{n_i}^+ \\ |\alpha_1| + \cdots + |\alpha_k| \leq j}} c_{\alpha_1,\ldots,\alpha_k} r^{2(|\alpha_1| + \cdots + |\alpha_k|)} \left\langle T_{1,\alpha_1} \cdots T_{k,\alpha_k} T_{k,\alpha_k}^* \cdots T_{1,\alpha_1}^* x, x \right\rangle < \epsilon$$

for any $j \geq N_0$ and $r \in (\delta, 1)$. This can be used to show that

$$\Phi^{q_1}_{f_1,T_1} \cdots \Phi^{q_k}_{f_k,T_k}(I) = \text{WOT-}\lim_{r\to 1} \Phi^{q_1}_{f_1,rT_1} \cdots \Phi^{q_k}_{f_k,rT_k}(I).$$

Hence, we deduce that $\mathbf{\Delta}^{\mathbf{p}}_{\mathbf{f},\mathbf{T}}(I) = \text{WOT-}\lim_{r\to 1} \mathbf{\Delta}^{\mathbf{p}}_{\mathbf{f},r\mathbf{T}}(I) \geq 0$ for any $\mathbf{p} \in \mathbb{Z}^k_+$ with $\mathbf{p} \leq \mathbf{m}$. Consequently, $\mathbf{T} \in \mathbf{D}^{\mathbf{m}}_{\mathbf{f}}(\mathcal{H})$ and it has the radial property. This completes the proof. $\square$

As expected, the domain $\mathbf{D}^{\mathbf{m}}_{\mathbf{f}}(\mathcal{H})$ is called *radial* if any $\mathbf{T} \in \mathbf{D}^{\mathbf{m}}_{\mathbf{f}}(\mathcal{H})$ has the radial property.

Corollary 1.5. *The noncommutative polydomain $\mathbf{D}^{\mathbf{m}}_{\mathbf{f}}(\mathcal{H})$ is radial.*

In the particular case when $k = 1$, Theorem 1.4 shows that any noncommutative domain $\mathbf{D}^{m_1}_{f_1}(\mathcal{H})$, $m_1 \in \mathbb{N}$, is radial. An important consequence is the following

Corollary 1.6. *All the results from* [42], [48], [49]*, which were proved in the setting of the radial part of $\mathbf{D}^{m_1}_{f_1}(\mathcal{H})$, are true for any domain $\mathbf{D}^{m_1}_{f_1}(\mathcal{H})$.*

Another consequence of Theorem 1.4 is the following

Corollary 1.7. *The following statements hold:*

(i) *If $\mathbf{f} = (f_1, f_2)$, and $\mathbf{T} = (T_1, T_2) \in \mathbf{D}^{m_1}_{f_1}(\mathcal{H}) \times_c \mathbf{D}^{m_2}_{f_2}(\mathcal{H})$ with $\mathbf{\Delta}^{\mathbf{m}}_{\mathbf{f},\mathbf{T}}(I) \geq 0$, then $\mathbf{T} \in \mathbf{D}^{\mathbf{m}}_{\mathbf{f}}(\mathcal{H})$.*

(ii) *If $\mathbf{T} = (T_1, \ldots, T_k) \in B(\mathcal{H})^{n_1} \times_c \cdots \times_c B(\mathcal{H})^{n_k}$ and $\Phi_{f_i,T_i}(I) = I$, $i \in \{1, \ldots, k\}$, then $\mathbf{T}$ is in the polydomain $\mathbf{D}^{\mathbf{m}}_{\mathbf{f}}(\mathcal{H})$.*

We say that a k-tuple $\mathbf{T} = (T_1, \ldots, T_k) \in \mathbf{D}^{\mathbf{m}}_{\mathbf{f}}(\mathcal{H})$ is *pure* if

$$\lim_{\mathbf{q}=(q_1,\ldots,q_k)\in\mathbb{Z}^k_+} (id - \Phi^{q_k}_{f_k,T_k}) \cdots (id - \Phi^{q_1}_{f_1,T_1})(I) = I.$$

We remark that $\{(id - \Phi^{q_k}_{f_k,T_k}) \cdots (id - \Phi^{q_1}_{f_1,T_1})(I)\}_{\mathbf{q}=(q_1,\ldots,q_k)\in\mathbb{Z}^k_+}$ is an increasing sequence of positive operators. Indeed, due to Theorem 1.4, $(id - \Phi_{f_k,T_k}) \cdots (id - \Phi_{f_1,T_1})(I) \geq 0$. Taking into account that $\Phi_{f_1,T_1}, \ldots, \Phi_{f_k,T_k}$ are commuting, we have

$$(id - \Phi^{q_k}_{f_k,T_k}) \cdots (id - \Phi^{q_1}_{f_1,T_1})(I) = \sum_{s=0}^{q_k-1} \Phi^s_{f_k,T_k} \cdots \sum_{s=0}^{q_1-1} \Phi^s_{f_1,T_1}(id - \Phi_{f_k,T_k}) \cdots (id - \Phi_{f_1,T_1})(I),$$

which proves our assertion. Note also that

$$(id - \Phi^{q_k}_{f_k,T_k}) \cdots (id - \Phi^{q_1}_{f_1,T_1})(I) \leq (id - \Phi^{q_{k-1}}_{f_{k-1},T_{k-1}}) \cdots (id - \Phi^{q_1}_{f_1,T_1})(I) \leq \cdots \leq (id - \Phi^{q_1}_{f_1,T_1})(I) \leq I.$$

Hence, we can deduce the following result.

Proposition 1.8. *A k-tuple $\mathbf{T} = (T_1, \ldots, T_k) \in \mathbf{D}^{\mathbf{m}}_{\mathbf{f}}(\mathcal{H})$ is pure if and only if, for each $i \in \{1, \ldots, k\}$, $\Phi^p_{f_i,T_i}(I) \to 0$ strongly as $p \to \infty$.*

A k-tuple $\mathbf{T} \in \mathbf{D}^{\mathbf{m}}_{\mathbf{f}}(\mathcal{H})$ is called *doubly commuting* if $T_{i,p}T^*_{j,q} = T^*_{j,q}T_{i,p}$ for any $i, j \in \{1, \ldots, k\}$ with $i \neq j$, and $p \in \{1, \ldots, n_i\}$, $q \in \{1, \ldots, n_j\}$. The next results provides some classes of elements in $\mathbf{D}^{\mathbf{m}}_{\mathbf{f}}(\mathcal{H})$.

Proposition 1.9. *Let $\mathbf{T} = (T_1, \ldots, T_k) \in B(\mathcal{H})^{n_1} \times_c \cdots \times_c B(\mathcal{H})^{n_k}$ be such that $\Phi_{f_i,T_i}(I) \leq I$ and let $\mathbf{m} = (m_1, \ldots, m_k) \in \mathbb{N}^k$. Then the following statements hold.*

(i) *If $\mathbf{\Delta}^{\mathbf{m}}_{\mathbf{f},\mathbf{T}}(I) \geq 0$ and, for each $i \in \{1, \ldots, k\}$, $\Phi^p_{f_i,T_i}(I) \to 0$ strongly as $p \to \infty$, then $\mathbf{T} \in \mathbf{D}^{\mathbf{m}}_{\mathbf{f}}(\mathcal{H})$.*

(ii) *If $\mathbf{T} \in \mathbf{D}^{m_1}_{f_1}(\mathcal{H}) \times_c \cdots \times_c \mathbf{D}^{m_k}_{f_k}(\mathcal{H})$ is doubly commuting, then $\mathbf{T} \in \mathbf{D}^{\mathbf{m}}_{\mathbf{f}}(\mathcal{H})$.*

(iii) *If $m_1\Phi_{f_1,T_1}(I) + \cdots + m_k\Phi_{f_k,T_k}(I) \leq I$, then $\mathbf{T} \in \mathbf{D}^{\mathbf{m}}_{\mathbf{f}}(\mathcal{H})$.*

(iv) *If $\mathbf{T} = (T_1, \ldots, T_k) \in \mathbf{D}^{\mathbf{m}}_{\mathbf{f}}(\mathcal{H})$, then $\mathbf{\Delta}^{\mathbf{m}}_{\mathbf{f},\mathbf{T}}(I) = 0$ if and only if*

$$(id - \Phi_{f_1,T_1}) \cdots (id - \Phi_{f_k,T_k})(I) = 0.$$

Proof. Applying Proposition 1.1 and Proposition 1.3, when $\Phi = (\Phi_{f_1,T_1}, \dots, \Phi_{f_k,T_k})$, we deduce part (i). To prove part (ii), note that since $T_i \in \mathbf{D}_{f_i}^{m_i}(\mathcal{H})$, we have $(id - \Phi_{f_i,T_i})^{p_i}(I) \geq 0$ for any $p_i \in \{0, 1, \dots, m_i\}$. Using the fact that $\mathbf{T}$ is doubly commuting, we deduce that

$$\mathbf{\Delta}_{\mathbf{f},\mathbf{T}}^{\mathbf{p}}(I) = (id - \Phi_{f_1,T_1})^{p_1}(I) \cdots (id - \Phi_{f_k,T_k})^{p_k}(I) \geq 0$$

for any $\mathbf{p} \in \mathbb{Z}_+^k$ with $\mathbf{p} \leq \mathbf{m}$, which shows that $\mathbf{T} \in \mathbf{D}_{\mathbf{f}}^{\mathbf{m}}(\mathcal{H})$.

Now, we prove part (iii). Let $p := m_1 + \cdots + m_k$ and set $i_j := 1$ if $1 \leq j \leq m_1$, $i_j := 2$ if $m_1 + 1 \leq j \leq m_1 + m_2, \dots$, and $i_j := k$ if $m_1 + \cdots + m_{k-1} + 1 \leq j \leq m_1 + \cdots + m_k$. Due to Theorem 1.4, to prove (iii) is equivalent to showing that if $\sum_{j=1}^{p} \Phi_{f_{i_j},T_{i_j}}(I) \leq I$, then

$$(id - \Phi_{f_{i_1},T_{i_1}}) \cdots (id - \Phi_{f_{i_p},T_{i_p}})(I) \geq 0.$$

Set $Y_{i_0} = I$ and $Y_{i_j} := (id - \Phi_{f_{i_j},T_{i_j}})(Y_{i_{j-1}})$ if $j \in \{1, \dots, p\}$. We proceed inductively. Note that $I = Y_{i_0} \geq Y_{i_1} = (id - \Phi_{f_{i_1},T_{i_1}})(I) \geq 0$. Let $n < p$ and assume that

$$I \geq Y_{i_n} \geq (id - \Phi_{f_{i_1},T_{i_1}} - \cdots - \Phi_{f_{i_n},T_{i_n}})(I) \geq 0.$$

Hence, we deduce that

$$\begin{aligned} I \geq Y_{i_n} \geq Y_{i_{n+1}} &= Y_{i_n} - \Phi_{f_{i_{n+1}},T_{i_{n+1}}}(Y_{i_n}) \\ &\geq (id - \Phi_{f_{i_1},T_{i_1}} - \cdots - \Phi_{f_{i_n},T_{i_n}})(I) - \Phi_{f_{i_{n+1}},T_{i_{n+1}}}(I), \end{aligned}$$

which proves our assertion.

Now, we prove part (iv). If $\mathbf{T} = (T_1, \dots, T_k) \in \mathbf{D}_{\mathbf{f}}^{\mathbf{m}}(\mathcal{H})$, Theorem 1.4 implies that

$$(id - \Phi_{f_1,T_1})^{p_1} \cdots (id - \Phi_{f_k,T_k})^{p_k}(I) \geq 0$$

for any $p_i \in \{0, 1, \dots, m_i\}$ and $i \in \{1, \dots, k\}$. Due to Lemma 6.2 from [42], if $\varphi : B(\mathcal{H}) \to B(\mathcal{H})$ is a power bounded positive linear map such that $D \in B(\mathcal{H})$ is a positive operator with $(id - \varphi)(D) \geq 0$, and $\gamma \geq 1$, then

$$(id - \varphi)^{\gamma}(D) = 0 \quad \text{if and only if} \quad (id - \varphi)(D) = 0.$$

Applying this result in our setting when $\varphi = \Phi_{f_1,T_1}$, $\gamma = m_1$, and $D = (id - \Phi_{f_2,T_2})^{m_2} \cdots (id - \Phi_{f_k,T_k})^{m_k}(I) \geq 0$, we deduce that relation $\mathbf{\Delta}_{\mathbf{f},\mathbf{T}}^{\mathbf{m}}(I) = 0$ is equivalent to $(id - \Phi_{f_1,T_1})(D) = 0$. Due to the commutativity of $\Phi_{f_1,T_1}, \cdots, \Phi_{f_k,T_k}$, the latter equality is equivalent to $(id - \Phi_{f_2,T_2})^{m_2}(\Lambda) = 0$, where $\Lambda := (id - \Phi_{f_3,T_3})^{m_3} \cdots (id - \Phi_{f_k,T_k})^{m_k}(id - \Phi_{f_1,T_1})(I) \geq 0$. Applying again the result mentioned above, we deduce that the latter equality is equivalent to $(id - \Phi_{f_2,T_2})(\Lambda) = 0$. Continuing this process, we can complete the proof of part (iv). ☐

2. Noncommutative Berezin transforms and universal models

Noncommutative Berezin transforms are used to show that each polydomain $\mathbf{D}_{\mathbf{f}}^{\mathbf{m}}(\mathcal{H})$ has a universal model $\mathbf{W} = \{\mathbf{W}_{i,j}\}$ consisting of weighted shifts acting on a tensor product of full Fock spaces.

Let H_{n_i} be an n_i-dimensional complex Hilbert space with orthonormal basis $e_1^i, \dots, e_{n_i}^i$. We consider the full Fock space of H_{n_i} defined by

$$F^2(H_{n_i}) := \bigoplus_{p \geq 0} H_{n_i}^{\otimes p},$$

where $H_{n_i}^{\otimes 0} := \mathbb{C}1$ and $H_{n_i}^{\otimes p}$ is the (Hilbert) tensor product of p copies of H_{n_i}. Set $e_\alpha^i := e_{j_1}^i \otimes \cdots \otimes e_{j_p}^i$ if $\alpha = g_{j_1}^i \cdots g_{j_p}^i \in \mathbb{F}_{n_i}^+$ and $e_{g_0^i}^i := 1 \in \mathbb{C}$. It is clear that $\{e_\alpha^i : \alpha \in \mathbb{F}_{n_i}^+\}$ is an orthonormal basis of $F^2(H_{n_i})$.

Let $m_i, n_i \in \mathbb{N} := \{1, 2, \dots\}$, $i \in \{1, \dots, k\}$, and $j \in \{1, \dots, n_i\}$. We define the *weighted left creation operators* $W_{i,j} : F^2(H_{n_i}) \to F^2(H_{n_i})$, associated with the abstract noncommutative domain $\mathbf{D}_{f_i}^{m_i}$ by setting

$$W_{i,j} e_\alpha^i := \frac{\sqrt{b_{i,\alpha}^{(m_i)}}}{\sqrt{b_{i,g_j\alpha}^{(m_i)}}} e_{g_j\alpha}^i, \qquad \alpha \in \mathbb{F}_{n_i}^+, \tag{2.1}$$

where

$$b_{i,g_0}^{(m_i)} := 1 \quad \text{and} \quad b_{i,\alpha}^{(m_i)} := \sum_{p=1}^{|\alpha|} \sum_{\substack{\gamma_1,\ldots,\gamma_p \in \mathbb{F}_{n_i}^+ \\ \gamma_1\cdots\gamma_p=\alpha \\ |\gamma_1|\geq 1,\ldots,|\gamma_p|\geq 1}} a_{i,\gamma_1}\cdots a_{i,\gamma_p} \binom{p+m_i-1}{m_i-1} \tag{2.2}$$

for all $\alpha \in \mathbb{F}_{n_i}^+$ with $|\alpha| \geq 1$.

Lemma 2.1. *For each $i \in \{1,\ldots,k\}$ and $j \in \{1,\ldots,n_i\}$, we define the operator $\mathbf{W}_{i,j}$ acting on the tensor Hilbert space $F^2(H_{n_1}) \otimes \cdots \otimes F^2(H_{n_k})$ by setting*

$$\mathbf{W}_{i,j} := \underbrace{I \otimes \cdots \otimes I}_{i-1 \text{ times}} \otimes W_{i,j} \otimes \underbrace{I \otimes \cdots \otimes I}_{k-i \text{ times}},$$

where the operators $W_{i,j}$ are defined by relation (2.1). If $\mathbf{W}_i := (\mathbf{W}_{i,1},\ldots,\mathbf{W}_{i,n_i})$, then the following statements hold.

(i) $(id - \Phi_{f_1,\mathbf{W}_1})^{m_1}\cdots(id - \Phi_{f_k,\mathbf{W}_k})^{m_k}(I) = \mathbf{P}_\mathbb{C}$, *where $\mathbf{P}_\mathbb{C}$ is the orthogonal projection from $\otimes_{i=1}^k F^2(H_{n_i})$ onto $\mathbb{C}1 \subset \otimes_{i=1}^k F^2(H_{n_i})$, where $\mathbb{C}1$ is identified with $\mathbb{C}1 \otimes \cdots \otimes \mathbb{C}1$.*
(ii) $\mathbf{W} := (\mathbf{W}_1,\ldots,\mathbf{W}_k)$ *is a pure k-tuple in the noncommutative polydomain $\mathbf{D}_\mathbf{f}^\mathbf{m}(\otimes_{i=1}^k F^2(H_{n_i}))$.*

Proof. Note that, due to relation (2.1), for each $i \in \{1,\ldots,k\}$ and $\beta_i \in \mathbb{F}_{n_i}^+$, we have

$$W_{i,\beta_i} W_{i,\beta_i}^* e_{\alpha_i}^i = \begin{cases} \frac{b_{i,\gamma_i}^{(m_i)}}{b_{i,\alpha_i}^{(m_i)}} e_{\alpha_i}^i & \text{if } \alpha_i = \beta_i\gamma_i, \ \gamma_i \in \mathbb{F}_{n_i}^+ \\ 0 & \text{otherwise.} \end{cases}$$

As in Lemma 1.2 from [42], straightforward computations reveal that $(id - \Phi_{f_i,\mathbf{W}_i})^{m_i}(I) = I \otimes \cdots \otimes I \otimes P_\mathbb{C} \otimes I \otimes \cdots \otimes I$, where $P_\mathbb{C}$ is on the i^{th} position and denotes the orthogonal projection from $F^2(H_{n_i})$ onto $\mathbb{C}1 \subset F^2(H_{n_i})$. Since the k-tuple $\mathbf{W} := (\mathbf{W}_1,\ldots,\mathbf{W}_k)$ is doubly commuting, we deduce that

$$(id - \Phi_{f_1,\mathbf{W}_1})^{m_1}\cdots(id - \Phi_{f_k,\mathbf{W}_k})^{m_k}(I) = (id - \Phi_{f_1,\mathbf{W}_1})^{m_1}(I)\cdots(id - \Phi_{f_k,\mathbf{W}_k})^{m_k}(I) = \mathbf{P}_\mathbb{C}.$$

which proves part (i). To prove part (ii), note first that relation (2.1) implies $\Phi_{f_i,W_i}^p(I)e_\alpha^i = 0$ if $p > |\alpha|$, $\alpha \in \mathbb{F}_{n_i}^+$. Since $\|\Phi_{f_i,\mathbf{W}_i}^p(I)\| \leq 1$ for any $p \in \mathbb{N}$, we deduce that $\lim_{p\to\infty} \Phi_{f_i,\mathbf{W}_i}^p(I) = 0$ in the strong operator topology. Taking into account that $\mathbf{\Delta}_{\mathbf{f},\mathbf{W}}^\mathbf{m}(I) = \mathbf{P}_\mathbb{C}$, we can use Proposition 1.3 to conclude that $\mathbf{W}$ is in the noncommutative polydomain $\mathbf{D}_\mathbf{f}^\mathbf{m}(\otimes_{i=1}^k F^2(H_{n_i}))$. Moreover, due to Proposition 1.8, $\mathbf{W}$ is a pure k-tuple in $\mathbf{D}_\mathbf{f}^\mathbf{m}(\otimes_{i=1}^k F^2(H_{n_i}))$. □

We mention that one can define the *weighted right creation operators* $\Lambda_{i,j} : F^2(H_{n_i}) \to F^2(H_{n_i})$ by setting

$$\Lambda_{i,j} e_\alpha^i := \frac{\sqrt{b_{i,\alpha}^{(m_i)}}}{\sqrt{b_{i,\alpha g_j}^{(m_i)}}} e_{\alpha g_j}^i, \qquad \alpha \in \mathbb{F}_{n_i}^+.$$

As in Lemma 2.1, it turns out that $\mathbf{\Lambda} := (\mathbf{\Lambda}_1,\ldots,\mathbf{\Lambda}_k)$ is a pure k-tuple in the noncommutative polydomain $\mathbf{D}_{\widetilde{\mathbf{f}}}^\mathbf{m}(\otimes_{i=1}^k F^2(H_{n_i}))$, where $\widetilde{\mathbf{f}} = (\widetilde{f}_1,\ldots,\widetilde{f}_k)$ with $\widetilde{f}_i := \sum_{|\alpha|\geq 1} a_{i,\widetilde{\alpha}} Z_\alpha$ and $\widetilde{\alpha} = g_{j_p}^i \cdots g_{j_1}^i$ denotes the reverse of $\alpha = g_{j_1}^i \cdots g_{j_p}^i \in \mathbb{F}_{n_i}^+$.

Throughout this paper, the k-tuple $\mathbf{W} := (\mathbf{W}_1,\ldots,\mathbf{W}_k)$ of Lemma 2.1 will be called the *universal model* associated with the abstract noncommutative polydomain $\mathbf{D}_\mathbf{f}^\mathbf{m}$. We introduce the *noncommutative Berezin kernel* associated with any element $\mathbf{T} = \{T_{i,j}\}$ in the noncommutative polydomain $\mathbf{D}_\mathbf{f}^\mathbf{m}(\mathcal{H})$ as the operator

$$\mathbf{K}_{\mathbf{f},\mathbf{T}} : \mathcal{H} \to F^2(H_{n_1}) \otimes \cdots \otimes F^2(H_{n_k}) \otimes \overline{\mathbf{\Delta}_{\mathbf{f},\mathbf{T}}^\mathbf{m}(I)(\mathcal{H})}$$

defined by

$$\mathbf{K}_{\mathbf{f},\mathbf{T}} h := \sum_{\beta_i \in \mathbb{F}_{n_i}^+, i=1,\ldots,k} \sqrt{b_{1,\beta_1}^{(m_1)}} \cdots \sqrt{b_{k,\beta_k}^{(m_k)}} e_{\beta_1}^1 \otimes \cdots \otimes e_{\beta_k}^k \otimes \mathbf{\Delta}_{\mathbf{f},\mathbf{T}}^\mathbf{m}(I)^{1/2} T_{1,\beta_1}^* \cdots T_{k,\beta_k}^* h,$$

where the defect operator is defined by

$$\mathbf{\Delta}^{\mathbf{m}}_{\mathbf{f},\mathbf{T}}(I) := (id - \Phi_{f_1,T_1})^{m_1} \cdots (id - \Phi_{f_k,T_k})^{m_k}(I),$$

and the coefficients $b^{(m_1)}_{1,\beta_1}, \ldots, b^{(m_k)}_{k,\beta_k}$ are given by relation (2.2). The fact that $\mathbf{K}_{\mathbf{f},\mathbf{T}}$ is a well-defined bounded operator will be proved in the next theorem.

Theorem 2.2. *The noncommutative Berezin kernel associated with a k-tuple $\mathbf{T} = (T_1, \ldots, T_k)$ in the noncommutative polydomain $\mathbf{D}^{\mathbf{m}}_{\mathbf{f}}(\mathcal{H})$ has the following properties.*

(i) $\mathbf{K}_{\mathbf{f},\mathbf{T}}$ *is a contraction and*

$$\mathbf{K}^*_{\mathbf{f},\mathbf{T}}\mathbf{K}_{\mathbf{f},\mathbf{T}} = \lim_{q_k\to\infty} \cdots \lim_{q_1\to\infty} (id - \Phi^{q_k}_{f_k,T_k}) \cdots (id - \Phi^{q_1}_{f_1,T_1})(I),$$

where the limits are in the weak operator topology.

(ii) *If* $\mathbf{T}$ *is pure, then*

$$\mathbf{K}^*_{\mathbf{f},\mathbf{T}}\mathbf{K}_{\mathbf{f},\mathbf{T}} = I_{\mathcal{H}}.$$

(iii) *For any* $i \in \{1, \ldots, k\}$ *and* $j \in \{1, \ldots, n_i\}$,

$$\mathbf{K}_{\mathbf{f},\mathbf{T}} T^*_{i,j} = (\mathbf{W}^*_{i,j} \otimes I)\mathbf{K}_{\mathbf{f},\mathbf{T}}.$$

Proof. Let $\mathbf{T} = (T_1, \ldots, T_k)$ be in the noncommutative polydomain $\mathbf{D}^{\mathbf{m}}_{\mathbf{f}}(\mathcal{H})$ and let $X \in B(\mathcal{H})$ be a positive operator such that

$$\mathbf{\Delta}^{\mathbf{p}}_{\mathbf{f},\mathbf{T}}(X) := (id - \Phi_{f_1,T_1})^{p_1} \cdots (id - \Phi_{f_k,T_k})^{p_k}(X) \geq 0$$

for any $\mathbf{p} := (p_1, \ldots, p_k) \in \mathbb{Z}^k_+$ with $p_i \in \{0, 1, \ldots, m_i\}$ and $i \in \{1, \ldots, k\}$. Fix $i \in \{1, \ldots, k\}$ and assume that $1 \leq p_i \leq m_i$. Then, due to the commutativity of $\Phi_{f_1,T_1}, \ldots, \Phi_{f_k,T_k}$, we have

$$(id - \Phi_{f_i,T_i})\mathbf{\Delta}^{\mathbf{p}-\mathbf{e_i}}_{\mathbf{f},\mathbf{T}}(X) = \mathbf{\Delta}^{\mathbf{p}}_{\mathbf{f},\mathbf{T}}(X) \geq 0,$$

where $\{\mathbf{e}_i\}_{i=1}^k$ is the canonical basis in $\mathbb{C}^k$. Hence, and using Lemma 1.2, we have

$$0 \leq \Phi_{f_i,T_i}(\mathbf{\Delta}^{\mathbf{p}-\mathbf{e_i}}_{\mathbf{f},\mathbf{T}}(X)) \leq \mathbf{\Delta}^{\mathbf{p}-\mathbf{e_i}}_{\mathbf{f},\mathbf{T}}(X) \leq X,$$

which proves that $\{\Phi^s_{f_i,T_i}(\mathbf{\Delta}^{\mathbf{p}-\mathbf{e_i}}_{\mathbf{f},\mathbf{T}}(X))\}_{s=0}^\infty$ is a decreasing sequence of positive operators which is convergent in the weak operator topology. Since Φ_{f_i,T_i} is WOT-continuous on bounded sets and $\Phi_{f_1,T_1}, \ldots, \Phi_{f_k,T_k}$ are commuting, we deduce that

$$\lim_{s\to\infty} \Phi^s_{f_i,T_i}(\mathbf{\Delta}^{\mathbf{p}-\mathbf{e_i}}_{\mathbf{f},\mathbf{T}}(X)) = \mathbf{\Delta}^{\mathbf{p}-\mathbf{e_i}}_{\mathbf{f},\mathbf{T}}\left(\lim_{s\to\infty} \Phi^s_{f_i,T_i}(X)\right). \tag{2.3}$$

Then we have

$$\begin{aligned} D^{(1)}_i(\mathbf{\Delta}^{\mathbf{p}}_{\mathbf{f},\mathbf{T}}(X)) &:= \sum_{s=0}^\infty \Phi^s_{f_i,T_i}(\mathbf{\Delta}^{\mathbf{p}}_{\mathbf{f},\mathbf{T}}(X)) = \sum_{s=0}^\infty \Phi^s_{f_i,T_i}\left[\mathbf{\Delta}^{\mathbf{p}-\mathbf{e_i}}_{\mathbf{f},\mathbf{T}}(X) - \Phi_{f_i,T_i}(\mathbf{\Delta}^{\mathbf{p}-\mathbf{e_i}}_{\mathbf{f},\mathbf{T}}(X))\right] \\ &= \mathbf{\Delta}^{\mathbf{p}-\mathbf{e_i}}_{\mathbf{f},\mathbf{T}}(X) - \lim_{q_1\to\infty} \Phi^{q_1}_{f_i,T_i}(\mathbf{\Delta}^{\mathbf{p}-\mathbf{e_i}}_{\mathbf{f},\mathbf{T}}(X)) \leq \mathbf{\Delta}^{\mathbf{p}-\mathbf{e_i}}_{\mathbf{f},\mathbf{T}}(X) \leq X. \end{aligned}$$

Due to relation (2.3), we deduce that

$$0 \leq D^{(1)}_i(\mathbf{\Delta}^{\mathbf{p}}_{\mathbf{f},\mathbf{T}}(X)) = \mathbf{\Delta}^{\mathbf{p}-\mathbf{e_i}}_{\mathbf{f},\mathbf{T}}\left(X - \lim_{q_1\to\infty} \Phi^{q_1}_{f_i,T_i}(X)\right), \qquad \mathbf{p} \leq \mathbf{m}, 1 \leq p_i.$$

Define $D^{(j)}_i(\mathbf{\Delta}^{\mathbf{p}}_{\mathbf{f},\mathbf{T}}(X)) := \sum_{s=0}^\infty \Phi^s_{f_i,T_i}(D^{(j-1)}_i(\mathbf{\Delta}^{\mathbf{p}}_{\mathbf{f},\mathbf{T}}(X)))$, where $j = 2, \ldots p_i$. Inductively, we can prove that

$$0 \leq D^{(j)}_i(\mathbf{\Delta}^{\mathbf{p}}_{\mathbf{f},\mathbf{T}}(X)) = \mathbf{\Delta}^{\mathbf{p}-j\mathbf{e_i}}_{\mathbf{f},\mathbf{T}}\left(X - \lim_{q_j\to\infty} \Phi^{q_j}_{f_i,T_i}(X)\right) \leq \mathbf{\Delta}^{\mathbf{p}-j\mathbf{e_i}}_{\mathbf{f},\mathbf{T}}(X) \leq X, \qquad j \leq p_i. \tag{2.4}$$

Indeed, if $j \leq p_i - 1$ and setting $Y := X - \lim_{q_j\to\infty} \Phi_{f_i,T_i}^{q_j}(X)$, relation (2.4) implies

$$\begin{aligned}
D_i^{(j+1)}(\mathbf{\Delta}_{\mathbf{f},\mathbf{T}}^{\mathbf{p}}(X)) &= \lim_{q_{j+1}\to\infty} \sum_{s=0}^{q_{j+1}} \Phi_{f_i,T_i}^s \left[\mathbf{\Delta}_{\mathbf{f},\mathbf{T}}^{\mathbf{p}-j\mathbf{e_i}}(Y)\right] \\
&= \mathbf{\Delta}_{\mathbf{f},\mathbf{T}}^{\mathbf{p}-(j+1)\mathbf{e_i}} \left[Y - \lim_{q_{j+1}\to\infty} \Phi_{f_i,T_i}^{q_{j+1}}(Y)\right] \\
&= \mathbf{\Delta}_{\mathbf{f},\mathbf{T}}^{\mathbf{p}-(j+1)\mathbf{e_i}}(Y) - \mathbf{\Delta}_{\mathbf{f},\mathbf{T}}^{\mathbf{p}-(j+1)\mathbf{e_i}} \left(\lim_{q_{j+1}\to\infty} \Phi_{f_i,T_i}^{q_{j+1}}(Y)\right).
\end{aligned}$$

On the other hand, we have

$$\begin{aligned}
\lim_{q_{j+1}\to\infty} \Phi_{f_i,T_i}^{q_{j+1}}(Y) &= \lim_{q_{j+1}\to\infty} \Phi_{f_i,T_i}^{q_{j+1}} \left(X - \lim_{q_j\to\infty} \Phi_{f_i,T_i}^{q_j}(X)\right) \\
&= \lim_{q_{j+1}\to\infty} \Phi_{f_i,T_i}^{q_{j+1}}(X) - \lim_{q_{j+1}\to\infty} \lim_{q_j\to\infty} \Phi_{f_i,T_i}^{q_{j+1}} \left(\Phi_{f_i,T_i}^{q_j}(X)\right) = 0.
\end{aligned}$$

Combining these results, we obtain

$$D_i^{(j+1)}(\mathbf{\Delta}_{\mathbf{f},\mathbf{T}}^{\mathbf{p}}(X)) = \mathbf{\Delta}_{\mathbf{f},\mathbf{T}}^{\mathbf{p}-(j+1)\mathbf{e_i}} \left(X - \lim_{q_j\to\infty} \Phi_{f_i,T_i}^{q_j}(X)\right) \leq \mathbf{\Delta}_{\mathbf{f},\mathbf{T}}^{\mathbf{p}-(j+1)\mathbf{e_i}}(X) \leq X,$$

for any $\mathbf{p} := (p_1, \ldots, p_k) \in \mathbb{Z}_+^k$ with $\mathbf{p} \leq \mathbf{m}$ and $p_i \geq 1$, which proves our assertion. When $j = p_i$, relation (2.4) becomes

$$0 \leq D_i^{(p_i)}(\mathbf{\Delta}_{\mathbf{f},\mathbf{T}}^{\mathbf{p}}(X)) = \mathbf{\Delta}_{\mathbf{f},\mathbf{T}}^{\mathbf{p}-p_i\mathbf{e_i}} \left(X - \lim_{q\to\infty} \Phi_{f_i,T_i}^{q}(X)\right) \leq X.$$

On the other hand, taking into account that we can rearrange WOT-convergent series of positive operators, we deduce that, for each $d \in \mathbb{N}$,

$$\begin{aligned}
\Phi_{f_i,T_i}^d(\mathbf{\Delta}_{\mathbf{f},\mathbf{T}}^{\mathbf{p}}(X)) &= \sum_{\alpha_1\in\mathbb{F}_{n_i}^+, |\alpha_1|\geq 1} a_{i,\alpha_1} T_{i,\alpha_1} \left(\cdots \sum_{\alpha_d\in\mathbb{F}_{n_1}^+, |\alpha_d|\geq 1} a_{i,\alpha_d} T_{i,\alpha_d} \mathbf{\Delta}_{\mathbf{f},\mathbf{T}}^{\mathbf{p}}(X) T_{i,\alpha_d}^* \cdots\right) T_{i,\alpha_1}^* \\
&= \sum_{\gamma\in\mathbb{F}_{n_i}^+, |\gamma|\geq d} \sum_{\substack{\alpha_1,\ldots,\alpha_d\in\mathbb{F}_{n_i}^+ \\ \alpha_1\cdots\alpha_d=\gamma \\ |\alpha_1|\geq 1,\ldots,|\alpha_d|\geq 1}} a_{i,\alpha_1} \cdots a_{i,\alpha_d} T_{i,\gamma} \mathbf{\Delta}_{\mathbf{f},\mathbf{T}}^{\mathbf{p}}(X) T_{i,\gamma}^*
\end{aligned}$$

and

$$\begin{aligned}
D_i^{(1)}(\mathbf{\Delta}_{\mathbf{f},\mathbf{T}}^{\mathbf{p}}(X)) &= \sum_{s=0}^{\infty} \Phi_{f_i,T_i}^s(\mathbf{\Delta}_{\mathbf{f},\mathbf{T}}^{\mathbf{p}}(X)) \\
&= \mathbf{\Delta}_{\mathbf{f},\mathbf{T}}^{\mathbf{p}}(X) + \sum_{\gamma\in\mathbb{F}_{n_i}^+, |\gamma|\geq 1} \left(\sum_{d=1}^{|\gamma|} \sum_{\substack{\alpha_1,\ldots,\alpha_d\in\mathbb{F}_{n_i}^+ \\ \alpha_1\cdots\alpha_d=\gamma \\ |\alpha_1|\geq 1,\ldots,|\alpha_d|\geq 1}} a_{i,\alpha_1} \cdots a_{i,\alpha_d}\right) T_{i,\gamma} \mathbf{\Delta}_{\mathbf{f},\mathbf{T}}^{\mathbf{p}}(X) T_{i,\gamma}^*.
\end{aligned}$$

Since $D_i^{(j)}(\mathbf{\Delta}_{\mathbf{f},\mathbf{T}}^{\mathbf{p}}(X)) := \sum_{s=0}^\infty \Phi_{f_i,T_i}^s(D_i^{(j-1)}(\mathbf{\Delta}_{\mathbf{f},\mathbf{T}}^{\mathbf{p}}(X)))$ for $j = 2, \ldots, p_i$, using a combinatorial argument and rearranging WOT-convergent series of positive operators, one can prove by induction over p_i that

$$\begin{aligned}
D_i^{(p_i)}(\mathbf{\Delta}_{\mathbf{f},\mathbf{T}}^{\mathbf{p}}(X)) &= \mathbf{\Delta}_{\mathbf{f},\mathbf{T}}^{\mathbf{p}}(X) + \sum_{\alpha\in\mathbb{F}_{n_i}^+, |\alpha|\geq 1} \left(\sum_{p=1}^{|\alpha|} \sum_{\substack{\gamma_1,\ldots,\gamma_p\in\mathbb{F}_{n_i}^+ \\ \gamma_1\cdots\gamma_p=\alpha \\ |\gamma_1|\geq 1,\ldots,|\gamma_p|\geq 1}} a_{i,\gamma_1} \cdots a_{i,\gamma_p} \binom{p+p_i-1}{p_i-1}\right) T_{i,\alpha} \mathbf{\Delta}_{\mathbf{f},\mathbf{T}}^{\mathbf{p}}(X) T_{i,\alpha}^* \\
&= \sum_{\alpha\in\mathbb{F}_{n_i}^+} b_{i,\alpha}^{(p_i)} T_{i,\alpha} \mathbf{\Delta}_{\mathbf{f},\mathbf{T}}^{\mathbf{p}}(X) T_{i,\alpha}^*.
\end{aligned}$$

For each $i \in \{1,\dots,k\}$, let $\Omega_i \subset B(\mathcal{H})$ be the set of all $Y \in B(\mathcal{H})$, $Y \geq 0$, such that the series $\sum_{\beta_i \in \mathbb{F}_{n_i}^+} b_{i,\beta_i}^{(m_i)} T_{i,\beta_i} Y T_{i,\beta_i}^*$ is convergent in the weak operator topology, where

$$b_{i,g_0}^{(m_i)} := 1 \quad \text{ and } \quad b_{i,\alpha}^{(m_i)} := \sum_{p=1}^{|\alpha|} \sum_{\substack{\gamma_1,\dots,\gamma_p \in \mathbb{F}_{n_i}^+ \\ \gamma_1 \cdots \gamma_p = \alpha \\ |\gamma_1| \geq 1, \dots, |\gamma_p| \geq 1}} a_{i,\gamma_1} \cdots a_{i,\gamma_p} \binom{p+m_i-1}{m_i-1}$$

for all $\alpha \in \mathbb{F}_{n_i}^+$ with $|\alpha| \geq 1$. We define the map $\Psi_i : \Omega_i \to B(\mathcal{H})$ by setting

$$\Psi_i(Y) := \sum_{\beta_i \in \mathbb{F}_{n_i}^+} b_{i,\beta_i}^{(m_i)} T_{i,\beta_i} Y T_{i,\beta_i}^*.$$

Due to the results above, we have

$$\begin{aligned} 0 \leq \Psi_i(\mathbf{\Delta}_{\mathbf{f},\mathbf{T}}^{\mathbf{p}}(X)) &= D_i^{(m_i)}(\mathbf{\Delta}_{\mathbf{f},\mathbf{T}}^{\mathbf{p}}(X)) \\ &= \mathbf{\Delta}_{\mathbf{f},\mathbf{T}}^{\mathbf{p}-m_i\mathbf{e_i}} \left(id - \lim_{q_i \to \infty} \Phi_{f_i,T_i}^{q_i} \right)(X) \\ &\leq \mathbf{\Delta}_{\mathbf{f},\mathbf{T}}^{\mathbf{p}-m_i\mathbf{e_i}}(X) \leq X, \end{aligned} \tag{2.5}$$

for any $\mathbf{p} := (p_1,\dots,p_k) \in \mathbb{Z}_+^k$ with $\mathbf{p} \leq \mathbf{m}$ and $p_i = m_i$. Since $\mathbf{T} = (T_1,\dots,T_k)$ is in the noncommutative polydomain $\mathbf{D}_{\mathbf{f}}^{\mathbf{m}}(\mathcal{H})$, Theorem 1.4 implies

$$\mathbf{\Delta}_{\mathbf{f},\mathbf{T}}^{\mathbf{p}}(I) := (id - \Phi_{f_1,T_1})^{p_1} \cdots (id - \Phi_{f_k,T_k})^{p_k}(I) \geq 0$$

for any $\mathbf{p} := (p_1,\dots,p_k) \in \mathbb{Z}_+^k$ with $p_i \in \{0,1,\dots,m_i\}$ and $i \in \{1,\dots,k\}$. Applying relation (2.5) in the particular case when $i=1$, $p_1 = m_1$, and $X = I$, we have

$$0 \leq \Psi_1(\mathbf{\Delta}_{\mathbf{f},\mathbf{T}}^{\mathbf{p}'}(I)) = D_1^{(m_1)}(\mathbf{\Delta}_{\mathbf{f},\mathbf{T}}^{\mathbf{p}'}(I)) = \mathbf{\Delta}_{\mathbf{f},\mathbf{T}}^{\mathbf{p}'-m_1\mathbf{e_1}} \left(I - \lim_{q_1 \to \infty} \Phi_{f_1,T_1}^{q_1}(I) \right) \leq \mathbf{\Delta}_{\mathbf{f},\mathbf{T}}^{\mathbf{p}'-m_1\mathbf{e_1}}(I) \leq I$$

for any $\mathbf{p}' = (m_1,p_2,\dots,p_k)$ with $\mathbf{p}' \leq \mathbf{m}$. Hence and using again relation (2.5), when $i=2$, $\mathbf{p} = (0,m_2,p_3\dots,p_k)$, and $X = \lim_{q_1\to\infty}\left(id - \Phi_{f_1,T_1}^{q_1} \right)(I) \geq 0$, we obtain

$$\begin{aligned} 0 \leq \Psi_2(\Psi_1(\mathbf{\Delta}_{\mathbf{f},\mathbf{T}}^{\mathbf{p}''}(I))) &= \Psi_2\left(\mathbf{\Delta}_{\mathbf{f},\mathbf{T}}^{\mathbf{p}''-m_1\mathbf{e_1}} \left(I - \lim_{q_1\to\infty} \Phi_{f_1,T_1}^{q_1}(I) \right) \right) \\ &= \mathbf{\Delta}_{\mathbf{f},\mathbf{T}}^{\mathbf{p}''-m_1\mathbf{e_1}-m_2\mathbf{e_2}} \lim_{q_2\to\infty} \lim_{q_1\to\infty} \left(id - \Phi_{f_2,T_2}^{q_2} \right) \left(id - \Phi_{f_1,T_1}^{q_1} \right)(I) \\ &\leq \mathbf{\Delta}_{\mathbf{f},\mathbf{T}}^{\mathbf{p}''-m_1\mathbf{e_1}-m_2\mathbf{e_2}}(I) \leq I \end{aligned}$$

for any $\mathbf{p}'' = (m_1,m_2,p_3,\dots,p_k)$. Continuing this process, a repeated application of (2.5), leads to the relation

$$0 \leq (\Psi_k \circ \cdots \circ \Psi_1)(\mathbf{\Delta}_{\mathbf{f},\mathbf{T}}^{\mathbf{m}}(I)) = \lim_{q_k\to\infty} \dots \lim_{q_1\to\infty} (id - \Phi_{f_k,T_k}^{q_k}) \cdots (id - \Phi_{f_1,T_1}^{q_1})(I) \leq I,$$

where $\mathbf{m} = (m_1,\dots,m_k)$. To prove item (i), note that the results above imply

$$\begin{aligned} \|\mathbf{K}_{\mathbf{f},\mathbf{T}} h\|^2 &= \sum_{\beta_k \in \mathbb{F}_{n_k}} \cdots \sum_{\beta_1 \in \mathbb{F}_{n_1}} b_{1,\beta_1}^{(m_1)} \cdots b_{k,\beta_k}^{(m_k)} \left\langle T_{k,\beta_k} \cdots T_{1,\beta_1} \mathbf{\Delta}_{\mathbf{f},\mathbf{T}}^{\mathbf{m}}(I) T_{1,\beta_1}^* \cdots T_{k,\beta_k}^* h, h \right\rangle \\ &= \left\langle (\Psi_k \circ \cdots \circ \Psi_1)(\mathbf{\Delta}_{\mathbf{f},\mathbf{T}}^{\mathbf{m}}(I)) h, h \right\rangle \leq \|h\|^2 \end{aligned}$$

for any $h \in \mathcal{H}$, and

$$\mathbf{K}_{\mathbf{f},\mathbf{T}}^* \mathbf{K}_{\mathbf{f},\mathbf{T}} = \lim_{q_k\to\infty} \dots \lim_{q_1\to\infty} (id - \Phi_{f_k,T_k}^{q_k}) \cdots (id - \Phi_{f_1,T_1}^{q_1})(I).$$

Now, item (ii) is clear. To prove part (iii), note that

$$W_{i,j}^* e^i_{\beta_i} = \begin{cases} \frac{\sqrt{b^{(m_i)}_{i,\gamma_i}}}{\sqrt{b^{(m_i)}_{i,\beta_i}}} e^i_{\gamma_i} & \text{if } \beta_i = g^i_j \gamma_i, \ \ \gamma_i \in \mathbb{F}^+_{n_i} \\ 0 & \text{otherwise} \end{cases} \tag{2.6}$$

for any $\beta_i \in \mathbb{F}^+_{n_i}$ and $j \in \{1, \ldots, n_i\}$. Hence, and using the definition of the noncommutative Berezin kernel, we have

$$\begin{aligned}
&(\mathbf{W}^*_{i,j} \otimes I)\mathbf{K}_{\mathbf{f},\mathbf{T}} h \\
&= \sum_{\beta_p \in \mathbb{F}^+_{n_p}, p \in \{1,\ldots,k\}} \sqrt{b^{(m_1)}_{1,\beta_1}} \cdots \sqrt{b^{(m_k)}_{k,\beta_k}} e^1_{\beta_1} \otimes \cdots \otimes e^{i-1}_{\beta_{i-1}} \otimes W^*_{i,j} e^i_{\beta_i} \otimes e^{i+1}_{\beta_{i+1}} \otimes \cdots \otimes e^k_{\beta_k} \otimes \mathbf{\Delta}^{\mathbf{m}}_{\mathbf{f},\mathbf{T}}(I)^{1/2} T^*_{1,\beta_1} \cdots T^*_{k,\beta_k} h \\
&= \sum_{\substack{\beta_p \in \mathbb{F}^+_{n_p}, p \in \{1,\ldots,k\} \setminus \{i\} \\ \gamma_i \in \mathbb{F}_{n_i}}} \sqrt{b^{(m_1)}_{1,\beta_1}} \cdots \sqrt{b^{(m_i)}_{i,\gamma_i}} \cdots \sqrt{b^{(m_k)}_{k,\beta_k}} e^1_{\beta_1} \otimes \cdots \otimes e^{i-1}_{\beta_{i-1}} \otimes e^i_{\gamma_i} \otimes e^{i+1}_{\beta_{i+1}} \otimes \cdots \otimes e^k_{\beta_k} \\
&\qquad \otimes \mathbf{\Delta}^{\mathbf{m}}_{\mathbf{f},\mathbf{T}}(I)^{1/2} T^*_{1,\beta_1} \cdots T^*_{i-1,\beta_i} T^*_{i, g^i_j \gamma_i} T^*_{i+1,\beta_{i+1}} \cdots T^*_{k,\beta_k} h
\end{aligned}$$

for any $h \in \mathcal{H}$. Using the commutativity of the tuples $T_1, \ldots, T_k$, we deduce that

$$(\mathbf{W}^*_{i,j} \otimes I)\mathbf{K}_{\mathbf{f},\mathbf{T}} = \mathbf{K}_{\mathbf{f},\mathbf{T}} T^*_{i,j}$$

for any $i \in \{1, \ldots, k\}$ and $j \in \{1, \ldots, n_i\}$. The proof is complete. □

Remark 2.3. *If $\mathbf{\Delta}^{\mathbf{m}}_{\mathbf{f},\mathbf{T}}(I) = 0$, then $\mathbf{K}_{\mathbf{f},\mathbf{T}} = 0$ and $(id - \Phi_{f_k,T_k}) \cdots (id - \Phi_{f_1,T_1})(I) = 0$.*

We can define now the *noncommutative Berezin transform* at $\mathbf{T} \in \mathbf{D}^{\mathbf{m}}_{\mathbf{f}}(\mathcal{H})$ to be the mapping $\mathbf{B}_{\mathbf{T}} : B(\otimes_{i=1}^k F^2(H_{n_i})) \to B(\mathcal{H})$ given by

$$\mathbf{B}_{\mathbf{T}}[g] := \mathbf{K}^*_{\mathbf{f},\mathbf{T}} (g \otimes I_{\mathcal{H}}) \mathbf{K}_{\mathbf{f},\mathbf{T}}, \qquad g \in B(\otimes_{i=1}^k F^2(H_{n_i})).$$

We denote by $\mathcal{P}(\mathbf{W})$ the set of all polynomials $p(\mathbf{W}_{i,j})$ in the operators $\mathbf{W}_{i,j}$, $i \in \{1, \ldots, k\}$, $j \in \{1, \ldots, n_i\}$, and the identity. We introduce the polydomain algebra $\mathcal{A}(\mathbf{D}^{\mathbf{m}}_{\mathbf{f}})$ to be the norm closed algebra generated by $\mathbf{W}_{i,j}$ and the identity.

Theorem 2.4. *Let $\mathbf{T} = \{T_{i,j}\}$ be in the noncommutative polydomain $\mathbf{D}^{\mathbf{m}}_{\mathbf{f}}(\mathcal{H})$ and let*

$$\mathcal{S} := \overline{\text{span}} \{ p(\mathbf{W}_{i,j}) q(\mathbf{W}_{i,j})^* : \ p(\mathbf{W}_{i,j}), q(\mathbf{W}_{i,j}) \in \mathcal{P}(\mathbf{W}) \},$$

where the closure is in the operator norm. Then there is a unital completely contractive linear map $\Psi_{\mathbf{f},\mathbf{T}} : \mathcal{S} \to B(\mathcal{H})$ such that

$$\Psi_{\mathbf{f},\mathbf{T}}(g) = \lim_{r \to 1} \mathbf{B}_{r\mathbf{T}}[g], \qquad g \in \mathcal{S},$$

where the limit exists in the norm topology of $B(\mathcal{H})$, and

$$\Psi_{\mathbf{f},\mathbf{T}} \left(\sum_{\gamma=1}^s p_\gamma(\mathbf{W}_{i,j}) q_\gamma(\mathbf{W}_{i,j})^* \right) = \sum_{\gamma=1}^s p_\gamma(T_{i,j}) q_\gamma(T_{i,j})^*$$

for any $p_\gamma(\mathbf{W}_{i,j}), q_\gamma(\mathbf{W}_{i,j}) \in \mathcal{P}(\mathbf{W})$ and $s \in \mathbb{N}$. In particular, the restriction of $\Psi_{\mathbf{f},\mathbf{T}}$ to the polydomain algebra $\mathcal{A}(\mathbf{D}^{\mathbf{m}}_{\mathbf{f}})$ is a completely contractive homomorphism. If, in addition, $\mathbf{T}$ is a pure k-tuple, then

$$\lim_{r \to 1} \mathbf{B}_{r\mathbf{T}}[g] = \mathbf{B}_T[g], \qquad g \in \mathcal{S}.$$

Proof. According to Theorem 1.4, $r\mathbf{T} = (rT_1, \ldots, rT_k) \in \mathbf{D}^{\mathbf{m}}_{\mathbf{f}}(\mathcal{H})$ for any $r \in (0,1)$. Since we have $\Phi^n_{f_i, rT_i}(I) \leq r^n \Phi^n_{f_i,T_i}(I) \leq r^n I$ for any $n \in \mathbb{N}$, Proposition 1.8 shows that $r\mathbf{T}$ is a pure k-tuple in $\mathbf{D}^{\mathbf{m}}_{\mathbf{f}}(\mathcal{H})$. Using Theorem 2.2, we deduce that the noncommutative Berezin kernel $\mathbf{K}_{\mathbf{f},r\mathbf{T}}$ is an isometry and

$$\mathbf{K}^*_{\mathbf{f},r\mathbf{T}} \left[p(\mathbf{W}_{i,j}) q(\mathbf{W}_{i,j})^* \otimes I_{\mathcal{H}} \right] \mathbf{K}_{\mathbf{f},r\mathbf{T}} = p(rT_{i,j}) q(rT_{i,j})^*, \qquad p(\mathbf{W}_{i,j}), q(\mathbf{W}_{i,j}) \in \mathcal{P}(\mathbf{W}). \tag{2.7}$$

Hence, we obtain the von Neumann [57] type inequality

$$\left\| \sum_{\gamma=1}^{s} p_\gamma(rT_{i,j}) q_\gamma(rT_{i,j})^* \right\| \leq \left\| \sum_{\gamma=1}^{s} p_\gamma(\mathbf{W}_{i,j}) q_\gamma(\mathbf{W}_{i,j})^* \right\| \tag{2.8}$$

for any $p_\gamma(\mathbf{W}_{i,j}), q_\gamma(\mathbf{W}_{i,j}) \in \mathcal{P}(\mathbf{W})$, $s \in \mathbb{N}$, and $r \in [0,1]$. Fix $g \in \mathcal{S}$ and let $\{\chi_n(\mathbf{W}_{i,j}, \mathbf{W}_{i,j}^*)\}_{n=0}^\infty$ be a sequence of operators in the span of $\mathcal{P}(\mathbf{W})\mathcal{P}(\mathbf{W})^*$ which converges to g in norm, as $n \to \infty$. Define $\Psi_{\mathbf{f},\mathbf{T}}(g) := \lim_{n\to\infty} \chi_n(T_{i,j}, T_{i,j}^*)$. The inequality (2.8) shows that $\Psi_{\mathbf{f},\mathbf{T}}(g)$ is well-defined and $\|\Psi_{\mathbf{f},\mathbf{T}}(g)\| \leq \|g\|$. Using the matrix version of (2.7), we deduce that $\Psi_{\mathbf{f},\mathbf{T}}$ is a unital completely contractive linear map. Now we prove that $\Psi_{\mathbf{f},\mathbf{T}}(g) = \lim_{r\to 1} \mathbf{B}_{r\mathbf{T}}[g]$. Note that relation (2.7) implies

$$\chi_n(rT_i, rT_i^*) = \mathbf{K}_{\mathbf{f},r\mathbf{T}}^* (\chi_n(\mathbf{W}_{i,j}, \mathbf{W}_{i,j}^*) \otimes I_\mathcal{H}) \mathbf{K}_{\mathbf{f},r\mathbf{T}} = \mathbf{B}_{r\mathbf{T}}[\chi_n(\mathbf{W}_{i,j}, \mathbf{W}_{i,j}^*)]$$

for any $n \in \mathbb{N}$ and $r \in (0,1)$. Using the fact that $\Psi_{f,rT}(g) := \lim_{n\to\infty} \chi_n(rT_i, rT_i^*)$ exists in norm, we deduce that

$$\Psi_{\mathbf{f},r\mathbf{T}}(g) = \mathbf{K}_{\mathbf{f},r\mathbf{T}}^* (g \otimes I_\mathcal{H}) \mathbf{K}_{\mathbf{f},r\mathbf{T}} = \mathbf{B}_{r\mathbf{T}}[g]. \tag{2.9}$$

Given $\epsilon > 0$ let $s \in \mathbb{N}$ be such that $\|\chi_s(W_i, W_i^*) - g\| < \frac{\epsilon}{3}$. Due to the first part of the theorem, we have

$$\|\Psi_{\mathbf{f},r\mathbf{T}}(g) - \chi_s(rT_i, rT_i^*)\| \leq \|g - \chi_s(\mathbf{W}_i, \mathbf{W}_i^*)\| < \frac{\epsilon}{3}$$

for any $r \in [0,1]$. On the other hand, since $\chi_s(\mathbf{W}_i, \mathbf{W}_i^*)$ has a finite number of terms, there exists $\delta \in (0,1)$ such that

$$\|\chi_s(rT_i, rT_i^*) - \chi_s(T_i, T_i^*)\| < \frac{\epsilon}{3}$$

for any $r \in (\delta, 1)$. Now, using these inequalities and relation (2.9), we deduce that

$$\begin{aligned} \|\Psi_{\mathbf{f},\mathbf{T}}(g) - \mathbf{B}_{r\mathbf{T}}[g]\| &= \|\Psi_{\mathbf{f},\mathbf{T}}(g) - \Psi_{\mathbf{f},r\mathbf{T}}(g)\| \\ &\leq \|\Psi_{\mathbf{f},\mathbf{T}}(g) - \chi_s(T_i, T_i^*)\| + \|\chi_s(T_i, T_i^*) - \chi_s(rT_i, rT_i^*)\| \\ &\quad + \|\chi_s(rT_i, rT_i^*) - \Psi_{\mathbf{f},r\mathbf{T}}(g)\| < \epsilon \end{aligned}$$

for any $r \in (\delta, 1)$, which proves our assertion. Now, we assume that $\mathbf{T} = (T_1, \ldots, T_k)$ is a pure k-tuple in $\mathbf{D}_\mathbf{f}^\mathbf{m}(\mathcal{H})$. Due to Theorem 2.2, we have

$$\mathbf{B}_\mathbf{T}[\chi_n(\mathbf{W}_{i,j}, \mathbf{W}_{i,j}^*)] := \mathbf{K}_{\mathbf{f},\mathbf{T}}^* (\chi_n(\mathbf{W}_{i,j}, \mathbf{W}_{i,j}^*) \otimes I_\mathcal{H}) \mathbf{K}_{\mathbf{f},\mathbf{T}} = \chi_n(T_{i,j}, T_{i,j}^*).$$

Taking into account that $\{\chi_n(\mathbf{W}_{i,j}, \mathbf{W}_{i,j}^*)\}_{n=0}^\infty$ is a sequence of operators in the span of $\mathcal{P}(\mathbf{W})\mathcal{P}(\mathbf{W})^*$ which converges to g in norm, we conclude that

$$\mathbf{B}_\mathbf{T}[g] = \Psi_{\mathbf{f},\mathbf{T}}(g) = \lim_{r\to 1} \mathbf{B}_{r\mathbf{T}}[g], \qquad g \in \mathcal{S}.$$

This completes the proof. □

We remark that Theorem 2.4 shows that the noncommutative polydomain algebra $\mathcal{A}(\mathbf{D}_\mathbf{f}^\mathbf{m})$ is the universal algebra generated by the identity and a doubly commuting k-tuple in the abstract polydomain domain $\mathbf{D}_\mathbf{f}^\mathbf{m}$.

We denote by $C^*(\mathbf{W}_{i,j})$ the C^*-algebra generated by the operators $\mathbf{W}_{i,j}$, where $i \in \{1, \ldots, k\}$, $j \in \{1, \ldots, n_i\}$, and the identity.

Corollary 2.5. *Let $\mathbf{q} = (q_1, \ldots, q_k)$ be a k-tuple of positive regular noncommutative polynomials and let*

$$\mathbf{X} := (X_1, \ldots, X_k) \in B(\mathcal{H})^{n_1} \times \cdots \times B(\mathcal{H})^{n_k}.$$

Then $\mathbf{X}$ is in the noncommutative polydomain $\mathbf{D}_\mathbf{q}^\mathbf{m}(\mathcal{H})$ if and only if there exists a unital completely positive linear map $\Psi : C^*(\mathbf{W}_{i,j}) \to B(\mathcal{H})$ such that

$$\Psi_{\mathbf{q},\mathbf{T}}(p(\mathbf{W}_{i,j}) r(\mathbf{W}_{i,j})^*) = p(X_{i,j}) r(X_{i,j})^*, \qquad p(\mathbf{W}_{i,j}), r(\mathbf{W}_{i,j}) \in \mathcal{P}(\mathbf{W}),$$

where $\mathbf{W} := \{\mathbf{W}_{i,j}\}$ is the universal model associated with the abstract noncommutative polydomain $\mathbf{D}_\mathbf{q}^\mathbf{m}$.

Proof. The direct implication is due to Theorem 2.4 and Arveson's extension theorem [3]. For the converse, note that, due to Lemma 2.1, Proposition 1.8, and Proposition 1.3, we have

$$(I-\Phi_{q_1,X_1})^{p_1}\cdots(I-\Phi_{q_k,X_k})^{p_k}(I)=\Psi_{\mathbf{q},\mathbf{T}}\left[(I-\Phi_{q_1,\mathbf{W}_1})^{p_1}\cdots(I-\Phi_{q_k,\mathbf{W}_k})^{p_k}(I)\right]\geq 0$$

for any $p_i\in\{0,1,\ldots,m_i\}$ and $i\in\{1,\ldots,k\}$. Using now Theorem 1.4 we can complete the proof. $\square$

We remark that under the condition

$$\overline{\text{span}}\,\{p(\mathbf{W}_{i,j})r(\mathbf{W}_{i,j})^*):\ p(\mathbf{W}_{i,j}),r(\mathbf{W}_{i,j})\in\mathcal{P}(\mathbf{W})\}=C^*(\mathbf{W}_{i,j}),$$

Corollary 2.5 shows that $C^*(\mathbf{W}_{i,j})$ is the universal C^*-algebra generated by the identity and a doubly commuting k-tuple in the abstract polydomain domain $\mathbf{D_f^m}$. We remark that the condition above holds, for example, if $\mathbf{D_f^m}(\mathcal{H})$ is the noncommutative polyball $[B(\mathcal{H})^{n_1}]_1^-\times_c\cdots\times_c[B(\mathcal{H})^{n_k}]_1^-$.

3. Noncommutative Hardy algebras and functional calculus

We introduce the noncommutative Hardy algebra $F^\infty(\mathbf{D_f^m})$ and provide a WOT-continuous functional calculus for completely non-coisometric tuples in in the noncommutative polydomain $\mathbf{D_f^m}(\mathcal{H})$.

Let $\varphi(\mathbf{W}_{i,j})=\sum\limits_{\beta_1\in\mathbb{F}_{n_1}^+,\ldots,\beta_k\in\mathbb{F}_{n_k}^+}c_{\beta_1,\ldots,\beta_k}\mathbf{W}_{1,\beta_1}\cdots\mathbf{W}_{k,\beta_k}$ be a formal sum with $c_{\beta_1,\ldots,\beta_k}\in\mathbb{C}$ and such that

$$\sum_{\beta_1\in\mathbb{F}_{n_1}^+,\ldots,\beta_k\in\mathbb{F}_{n_k}^+}|c_{\beta_1,\ldots,\beta_k}|^2\frac{1}{b_{1,\beta_1}^{(m_1)}\cdots b_{k,\beta_k}^{(m_k)}}<\infty.$$

We prove that $\varphi(\mathbf{W}_{i,j})(e^1_{\gamma_1}\otimes\cdots\otimes e^k_{\gamma_k})$ is in $\otimes_{i=1}^kF^2(H_{n_i})$, for any $\gamma_1\in\mathbb{F}_{n_1}^+,\ldots,\gamma_k\in\mathbb{F}_{n_k}^+$. Indeed, due to relation (2.1), we have

$$\begin{aligned}\sum_{\beta_1\in\mathbb{F}_{n_1}^+,\ldots,\beta_k\in\mathbb{F}_{n_k}^+}&c_{\beta_1,\ldots,\beta_k}\mathbf{W}_{1,\beta_1}\cdots\mathbf{W}_{k,\beta_k}(e^1_{\gamma_1}\otimes\cdots\otimes e^k_{\gamma_k})\\&=\sum_{\beta_1\in\mathbb{F}_{n_1}^+,\ldots,\beta_k\in\mathbb{F}_{n_k}^+}c_{\beta_1,\ldots,\beta_k}\sqrt{\frac{b_{1,\gamma_1}^{(m_1)}}{b_{1,\beta_1\gamma_1}^{(m_1)}}}\cdots\sqrt{\frac{b_{k,\gamma_k}^{(m_k)}}{b_{k,\beta_k\gamma_k}^{(m_k)}}}e^1_{\beta_1\gamma_1}\otimes\cdots\otimes e^k_{\beta_k\gamma_k}.\end{aligned}$$

Let $i\in\{1,\ldots,k\}$ and $\alpha,\beta\in\mathbb{F}_{n_i}$ be such that $|\alpha|\geq 1$ and $|\beta|\geq 1$. Note that, for any $j\in\{1,\ldots,|\alpha|\}$ and $k\in\{1,\ldots,|\beta|\}$,

$$\binom{j+m_i-1}{m_i-1}\binom{k+m_i-1}{m_i-1}\leq C_{i,|\beta|}^{(m_i)}\binom{j+k+m_i-1}{m_i-1},$$

where $C_{i,|\beta|}^{(m_i)}:=\binom{|\beta|+m_i-1}{m_i-1}$. Using relation (2.2) and comparing the product $b_{i,\alpha}^{(m_i)}b_{i,\beta}^{(m_i)}$ with $b_{i,\alpha\beta}^{(m)}$, we deduce that

$$b_{i,\alpha}^{(m_i)}b_{i,\beta}^{(m_i)}\leq C_{i,|\beta|}^{(m_i)}b_{i,\alpha\beta}^{(m_i)}\quad\text{and}\quad b_{i,\alpha}^{(m_i)}b_{i,\beta}^{(m_i)}\leq C_{i,|\alpha|}^{(m_i)}b_{i,\alpha\beta}^{(m_i)}\tag{3.1}$$

for any $\alpha,\beta\in\mathbb{F}_{n_i}^+$. Hence, we deduce that

$$\begin{aligned}\sum_{\beta_1\in\mathbb{F}_{n_1}^+,\ldots,\beta_k\in\mathbb{F}_{n_k}^+}&|c_{\beta_1,\ldots,\beta_k}|^2\frac{b_{1,\gamma_1}^{(m_1)}}{b_{1,\beta_1\gamma_1}^{(m_1)}}\cdots\frac{b_{k,\gamma_k}^{(m_k)}}{b_{k,\beta_k\gamma_k}^{(m_k)}}\\&\leq C_{1,|\gamma_1|}^{(m_1)}\cdots C_{k,|\gamma_k|}^{(m_k)}\sum_{\beta_1\in\mathbb{F}_{n_1}^+,\ldots,\beta_k\in\mathbb{F}_{n_k}^+}|c_{\beta_1,\ldots,\beta_k}|^2\frac{1}{b_{1,\beta_1}^{(m_1)}\cdots b_{k,\beta_k}^{(m_k)}}<\infty,\end{aligned}$$

which proves our assertion. Let $\mathcal{P}$ be the linear span of the vectors $e_{\gamma_1}\otimes\cdots\otimes e_{\gamma_k}$ for $\gamma_1\in\mathbb{F}_{n_1}^+,\ldots,\gamma_k\in\mathbb{F}_{n_k}^+$. If

$$\sup_{p\in\mathcal{P},\|p\|\leq 1}\left\|\sum_{\beta_1\in\mathbb{F}_{n_1}^+,\ldots,\beta_k\in\mathbb{F}_{n_k}^+}c_{\beta_1,\ldots,\beta_k}\mathbf{W}_{1,\beta_1}\cdots\mathbf{W}_{k,\beta_k}(p)\right\|<\infty,$$

then there is a unique bounded operator acting on $F^2(H_{n_1})\otimes\cdots\otimes F^2(H_{n_k})$, which we denote by $\varphi(\mathbf{W}_{i,j})$, such that

$$\varphi(\mathbf{W}_{i,j})p=\sum_{\beta_1\in\mathbb{F}_{n_1}^+,\ldots,\beta_k\in\mathbb{F}_{n_k}^+} c_{\beta_1,\ldots,\beta_k}\mathbf{W}_{1,\beta_1}\cdots\mathbf{W}_{k,\beta_k}(p)\quad \text{for any}\quad p\in\mathcal{P}.$$

The set of all operators $\varphi(\mathbf{W}_{i,j})\in B(\otimes_{i=1}^k F^2(H_{n_i}))$ satisfying the above-mentioned properties is denoted by $F^\infty(\mathbf{D_f^m})$. One can prove that $F^\infty(\mathbf{D_f^m})$ is a Banach algebra, which we call Hardy algebra associated with the noncommutative polydomain $\mathbf{D_f^m}$.

In a similar manner, one can define the Hardy algebra $R^\infty(\mathbf{D_f^m})$. For each $i\in\{1,\ldots,k\}$ and $j\in\{1,\ldots,n_i\}$, we define the operator $\mathbf{\Lambda}_{i,j}$ acting on the Hilbert space $F^2(H_{n_1})\otimes\cdots\otimes F^2(H_{n_k})$ by setting

$$\mathbf{\Lambda}_{i,j}:=\underbrace{I\otimes\cdots\otimes I}_{i-1\text{ times}}\otimes\Lambda_{i,j}\otimes\underbrace{I\otimes\cdots\otimes I}_{k-i\text{ times}}.$$

Set $\mathbf{\Lambda}_i:=(\mathbf{\Lambda}_{i,1},\ldots,\mathbf{\Lambda}_{i,n_i})$. As in Lemma 2.1, one can prove that, $\mathbf{\Lambda}:=(\mathbf{\Lambda}_1,\ldots,\mathbf{\Lambda}_k)$ is in the noncommutative polydomain $\mathbf{D}^{\mathbf{m}}_{\tilde{\mathbf{f}}}(\otimes_{i=1}^k F^2(H_{n_i}))$, where $\tilde{\mathbf{f}}=(\tilde f_1,\ldots,\tilde f_k)$.

Let $\chi(\mathbf{\Lambda}_{i,j})=\sum_{\beta_1\in\mathbb{F}_{n_1}^+,\ldots,\beta_k\in\mathbb{F}_{n_k}^+} c_{\tilde\beta_1,\ldots,\tilde\beta_k}\mathbf{\Lambda}_{1,\beta_1}\cdots\mathbf{\Lambda}_{k,\beta_k}$ be a formal sum with $c_{\tilde\beta_1,\ldots,\tilde\beta_k}\in\mathbb{C}$ and such that

$$\sum_{\beta_1\in\mathbb{F}_{n_1}^+,\ldots,\beta_k\in\mathbb{F}_{n_k}^+}|c_{\beta_1,\ldots,\beta_k}|^2\frac{1}{b_{1,\beta_1}^{(m_1)}\cdots b_{k,\beta_k}^{(m_k)}}<\infty$$

and

$$\sup_{p\in\mathcal{P},\|p\|\le 1}\left\|\sum_{\beta_1\in\mathbb{F}_{n_1}^+,\ldots,\beta_k\in\mathbb{F}_{n_k}^+} c_{\tilde\beta_1,\ldots,\tilde\beta_k}\mathbf{\Lambda}_{1,\beta_1}\cdots\mathbf{\Lambda}_{k,\beta_k}(p)\right\|<\infty.$$

Then there is a unique bounded operator acting on $F^2(H_{n_1})\otimes\cdots\otimes F^2(H_{n_k})$, which we denote by $\chi(\mathbf{\Lambda}_{i,j})$, such that

$$\chi(\mathbf{\Lambda}_{i,j})p=\sum_{\beta_1\in\mathbb{F}_{n_1}^+,\ldots,\beta_k\in\mathbb{F}_{n_k}^+} c_{\tilde\beta_1,\ldots,\tilde\beta_k}\mathbf{\Lambda}_{1,\beta_1}\cdots\mathbf{\Lambda}_{k,\beta_k}(p)\quad \text{for any}\quad p\in\mathcal{P}.$$

The set of all operators $\chi(\mathbf{\Lambda}_{i,j})\in B(\otimes_{i=1}^k F^2(H_{n_i}))$ satisfying the above-mentioned properties is a Banach algebra which is denoted by $R^\infty(\mathbf{D_f^m})$.

Proposition 3.1. *The following statements hold:*

(i) $F^\infty(\mathbf{D_f^m})'=R^\infty(\mathbf{D_f^m})$, *where* $'$ *stands for the commutant;*
(ii) $F^\infty(\mathbf{D_f^m})''=F^\infty(\mathbf{D_f^m})$*;*
(iii) $F^\infty(\mathbf{D_f^m})$ *is WOT-closed in* $B(\otimes_{i=1}^k F^2(H_{n_i}))$*.*

Proof. Let $U\in B(\otimes_{i=1}^k F^2(H_{n_i}))$ be the unitary operator defined by equation

$$U(e^1_{\gamma_1}\otimes\cdots\otimes e^k_{\gamma_k}):=(e^1_{\tilde\gamma_1}\otimes\cdots\otimes e^k_{\tilde\gamma_k}),\qquad \gamma_1\in\mathbb{F}_{n_1}^+,\ldots,\gamma_k\in\mathbb{F}_{n_k}^+,$$

and note that $U^*\mathbf{\Lambda}_{i,j}U=\mathbf{W}^{\tilde{\mathbf{f}}}{}_{i,j}$ for any $i=1,\ldots,k$ and $j\in\{1,\ldots,n_i\}$, where $\mathbf{W}^{\tilde{\mathbf{f}}}{}_{i,j}$ is the universal model associated with $\mathbf{D}^{\mathbf{m}}_{\tilde{\mathbf{f}}}$. Consequently, we have $U^*(F^\infty(\mathbf{D}^{\mathbf{m}}_{\tilde{\mathbf{f}}}))U=R^\infty(\mathbf{D_f^m})$. On the other hand, since $\mathbf{W}_{i_1,j_1}\mathbf{\Lambda}_{i_2,j_2}=\mathbf{\Lambda}_{i_2,j_2}\mathbf{W}_{i_1,j_1}$ for any $i_1,i_2\in\{1,\ldots,k\}$, $j_1\in\{1,\ldots,n_{i_1}\}$, and $j_2\in\{1,\ldots,n_{i_2}\}$. we deduce that $R^\infty(\mathbf{D_f^m})\subseteq F^\infty(\mathbf{D_f^m})'$. Now, we prove the reverse inclusion. Let $G\in F^\infty(\mathbf{D_f^m})'$ and note that, since $G(1)\in\otimes_{i=1}^k F^2(H_{n_i})$, we have

$$G(1)=\sum_{\beta_1\in\mathbb{F}_{n_1}^+,\ldots,\beta_k\in\mathbb{F}_{n_k}^+} c_{\tilde\beta_1,\ldots,\tilde\beta_k}\frac{1}{\sqrt{b_{1,\tilde\beta_1}^{(m_1)}}}\cdots\frac{1}{\sqrt{b_{k,\tilde\beta_k}^{(m_k)}}}e^1_{\tilde\beta_1}\otimes\cdots\otimes e^k_{\tilde\beta_k}$$

for some coefficients $c_{\tilde\beta_1,\ldots,\tilde\beta_k}\in\mathbb{C}$ with

$$\sum_{\beta_1\in\mathbb{F}_{n_1}^+,\ldots,\beta_k\in\mathbb{F}_{n_k}^+}|c_{\beta_1,\ldots,\beta_k}|^2\frac{1}{b_{1,\beta_1}^{(m_1)}\cdots b_{k,\beta_k}^{(m_k)}}<\infty.$$

Taking into account that $G\mathbf{W}_{i,j} = \mathbf{W}_{i,j}G$ for $i \in \{1,\dots,k\}$ and $j \in \{1,\dots,n_i\}$, relations (2.6) and its analogue for $\mathbf{\Lambda}_{i,j}$ imply

$$\begin{aligned}
G(e^1_{\alpha_1}\otimes\cdots\otimes e^k_{\alpha_k}) &= \sqrt{b^{(m_1)}_{1,\alpha_1}}\cdots\sqrt{b^{(m_k)}_{k,\alpha_k}}\,GW_{1,\alpha_1}\cdots W_{k,\alpha_k}(1)\\
&= \sqrt{b^{(m_1)}_{1,\alpha_1}}\cdots\sqrt{b^{(m_k)}_{k,\alpha_k}}\,W_{1,\alpha_1}\cdots W_{k,\alpha_k}G(1)\\
&= \sum_{\beta_1\in\mathbb{F}^+_{n_1},\dots,\beta_k\in\mathbb{F}^+_{n_k}} c_{\tilde\beta_1,\dots,\tilde\beta_k}\frac{\sqrt{b^{(m_1)}_{1,\alpha_1}}}{\sqrt{b^{(m_1)}_{1,\alpha_1\tilde\beta_1}}}\cdots\frac{\sqrt{b^{(m_k)}_{k,\alpha_k}}}{\sqrt{b^{(m_k)}_{k,\alpha_k\tilde\beta_k}}}e^1_{\alpha_1\tilde\beta_1}\otimes\cdots\otimes e^k_{\alpha_k\tilde\beta_k}\\
&= \sum_{\beta_1\in\mathbb{F}^+_{n_1},\dots,\beta_k\in\mathbb{F}^+_{n_k}} c_{\tilde\beta_1,\dots,\tilde\beta_k}\mathbf{\Lambda}_{1,\beta_1}\cdots\mathbf{\Lambda}_{k,\beta_k}(e^1_{\alpha_1}\otimes\cdots\otimes e^k_{\alpha_k})
\end{aligned}$$

for any $\alpha_1\in\mathbb{F}^+_{n_1},\dots,\alpha_k\in\mathbb{F}^+_{n_k}$. Therefore,

$$G(p) = \sum_{\beta_1\in\mathbb{F}^+_{n_1},\dots,\beta_k\in\mathbb{F}^+_{n_k}} c_{\tilde\beta_1,\dots,\tilde\beta_k}\mathbf{\Lambda}_{1,\beta_1}\cdots\mathbf{\Lambda}_{k,\beta_k}(p)$$

for any polynomial for any $p\in\mathcal{P}$. Since G is a bounded operator,

$$g(\mathbf{\Lambda}_{i,j}) := \sum_{\beta_1\in\mathbb{F}^+_{n_1},\dots,\beta_k\in\mathbb{F}^+_{n_k}} c_{\tilde\beta_1,\dots,\tilde\beta_k}\mathbf{\Lambda}_{1,\beta_1}\cdots\mathbf{\Lambda}_{k,\beta_k}$$

is in $R^\infty(\mathbf{D^m_f})$ and $G = g(\mathbf{\Lambda}_{i,j})$. Therefore, $R^\infty(\mathbf{D^m_f}) = F^\infty(\mathbf{D^m_f})'$. The item (ii) follows easily applying part (i). Now, item (iii) is clear. This completes the proof. $\square$

Similarly to the proof of Proposition 3.1, one can prove that if $\mathcal{S}\subset B(\mathcal{K})$ and $I_\mathcal{K}\in\mathcal{S}$, then

$$(F^\infty(\mathbf{D^m_f})\otimes\mathcal{S})' = R^\infty(\mathbf{D^m_f})\bar\otimes\mathcal{S}' \quad\text{ and }\quad (R^\infty(\mathbf{D^m_f})\otimes\mathcal{S})' = F^\infty(\mathbf{D^m_f})\bar\otimes\mathcal{S}',$$

where $F^\infty(\mathbf{D^m_f})\bar\otimes\mathcal{S}'$ is the WOT-closed algebra generated by the spatial tensor product of the two algebras. Moreover, for each $i\in\{1,\dots,k\}$, the commutant of the set

$$\{W_{i,j}\otimes I_\mathcal{H}:\ j\in\{1,\dots,n_i\}\}\cup\{I_{F^2(H_{n_i}}\otimes Y:\ Y\in\mathcal{S}\}$$

is equal to $R^\infty(\mathbf{D}^{m_1}_{f_1})\bar\otimes\mathcal{S}'$. A repeated appplication of these results shows that, if $\mathbf{f}=(f_1,\dots,f_k)$ and $\mathbf{m}=(m_1,\dots,m_k)$, then

$$F^\infty(\mathbf{D^m_f})\bar\otimes B(\mathcal{H}) = F^\infty(\mathbf{D}^{m_1}_{f_1})\bar\otimes\cdots\bar\otimes F^\infty(\mathbf{D}^{m_k}_{f_k})\bar\otimes B(\mathcal{H})$$

In the same manner, one can prove the corresponding result for $R^\infty(\mathbf{D^m_f})\bar\otimes B(\mathcal{H})$. Another consequence of the results above is the following Tomita-type theorem in our non-selfadjoint setting: if $\mathcal{M}$ is a von Neumann algebra, then

$$(F^\infty(\mathbf{D^m_f})\bar\otimes\mathcal{M})'' = F^\infty(\mathbf{D^m_f})\bar\otimes\mathcal{M}.$$

Proposition 3.2. *The noncommutative Hardy algebra $F^\infty(\mathbf{D^m_f})$ is the sequential SOT-(resp. WOT-, w^*-) closure of all polynomials in $\mathbf{W}_{i,j}$ and the identity, where $i\in\{1,\dots,k\}$, $j\in\{1,\dots,n_k\}$.*

Proof. Let P_n, $n\ge 0$, be the orthogonal projection of $F^2(H_{n_1})\otimes\cdots\otimes F^2(H_{n_k})$ on the the subspace span$\{e_{\alpha_1}\otimes\cdots\otimes e_{\alpha_k}:\ |\alpha_1|+\cdots+|\alpha_k|=n,\ \alpha_1\in\mathbb{F}^+_{n_1},\dots,\alpha_k\in\mathbb{F}^+_{n_k}\}$. Define the completely contractive projection $\Gamma_j: B(\otimes_{i=1}^k F^2(H_{n_i}))\to B(\otimes_{i=1}^k F^2(H_{n_i}))$, $j\in\mathbb{Z}$, by

$$\Gamma_j(A) := \sum_{n\ge\max\{0,-j\}} P_nAP_{n+j}.$$

The Cesaro operators on $B(\otimes_{i=1}^k F^2(H_{n_i}))$, defined by

$$\chi_n(A) := \sum_{|j|<n}\left(1-\frac{|j|}{n}\right)\Gamma_j(A),\quad n\ge 1,$$

are completely contractive and $\chi_n(A)$ converges to A in the strong operator topology. Let $A\in F^\infty(\mathbf{D}_\mathbf{f}^\mathbf{m})$ have the Fourier representation $\sum_{\beta_1\in\mathbb{F}_{n_1}^+,\ldots,\beta_k\in\mathbb{F}_{n_k}^+} c_{\beta_1,\ldots,\beta_k}\mathbf{W}_{1,\beta_1}\cdots\mathbf{W}_{k,\beta_k}$. Taking into account the definition of the operators $\mathbf{W}_{i,j}$, one can easily check that

$$P_{n+j}AP_j=\left(\sum_{\substack{|\beta_1|+\cdots|\beta_k|=n\\ \beta_1\in\mathbb{F}_{n_1}^+,\ldots,\beta_k\in\mathbb{F}_{n_k}^+}} c_{\beta_1,\ldots,\beta_k}\mathbf{W}_{1,\beta_1}\cdots\mathbf{W}_{k,\beta_k}\right)P_j,\quad n\geq 0, j\geq 0,$$

and $P_jAP_{n+j}=0$ if $n\geq 1$ and $j\geq 0$. Therefore,

$$\chi_k(A)=\sum_{0\leq q\leq n-1}\left(1-\frac{q}{n}\right)\left(\sum_{\substack{|\beta_1|+\cdots|\beta_k|=q\\ \beta_1\in\mathbb{F}_{n_1}^+,\ldots,\beta_k\in\mathbb{F}_{n_k}^+}} c_{\beta_1,\ldots,\beta_k}\mathbf{W}_{1,\beta_1}\cdots\mathbf{W}_{k,\beta_k}\right)$$

converges to A, as $k\to\infty$, in the strong operator topology. The proof is complete. $\square$

Lemma 3.3. *Let $\mathbf{W}:=(\mathbf{W}_1,\ldots,\mathbf{W}_k)$ be the universal model associated to the abstract noncommutative domain $\mathbf{D}_\mathbf{f}^\mathbf{m}$, where $\mathbf{W}_i:=(\mathbf{W}_{i,1},\ldots,\mathbf{W}_{i,n_i})$ for $i\in\{1,\ldots,k\}$. If*

$$\varphi(\mathbf{W}_{i,j})=\sum_{\beta_1\in\mathbb{F}_{n_1}^+,\ldots,\beta_k\in\mathbb{F}_{n_k}^+} c_{\beta_1,\ldots,\beta_k}\mathbf{W}_{1,\beta_1}\cdots\mathbf{W}_{k,\beta_k}$$

is in the noncommutative Hardy algebra $F^\infty(\mathbf{D}_\mathbf{f}^\mathbf{m})$, then the following statements hold.

(i) *The series*

$$\varphi(r\mathbf{W}_{i,j}):=\sum_{q=0}^\infty\sum_{\substack{\beta_1\in\mathbb{F}_{n_1}^+,\ldots,\beta_k\in\mathbb{F}_{n_k}^+\\ |\beta_1|+\cdots+|\beta_k|=q}} r^q c_{\beta_1,\ldots,\beta_k}\mathbf{W}_{1,\beta_1}\cdots\mathbf{W}_{k,\beta_k}$$

converges in the operator norm topology for any $r\in[0,1)$.

(ii) *The operator $\varphi(r\mathbf{W}_{i,j})$ is in the noncommutative domain algebra $\mathcal{A}(\mathbf{D}_\mathbf{f}^\mathbf{m})$ and*

$$\|\varphi(r\mathbf{W}_{i,j})\|\leq\|\varphi(\mathbf{W}_{i,j})\|.$$

(iii) $\varphi(\mathbf{W}_{i,j})=\text{SOT-}\lim_{r\to 1}\varphi(r\mathbf{W}_{i,j})$ *and*

$$\|\varphi(\mathbf{W}_{i,j})\|=\sup_{0\leq r<1}\|\varphi(r\mathbf{W}_{i,j})\|=\lim_{r\to 1}\|\varphi(r\mathbf{W}_{i,j})\|.$$

Proof. First, we prove that

$$\sum_{\substack{\beta_1\in\mathbb{F}_{n_1}^+,\ldots,\beta_k\in\mathbb{F}_{n_k}^+\\ |\beta_1|=p_1,\ldots,|\beta_k|=p_k}} b_{1,\beta_1}^{(m_1)}\cdots b_{k,\beta_k}^{(m_k)}\mathbf{W}_{1,\beta_1}\cdots\mathbf{W}_{k,\beta_k}\mathbf{W}_{k,\beta_k}^*\cdots\mathbf{W}_{1,\beta_1}^*$$
$$\leq\binom{p_1+m_1-1}{m_1-1}\cdots\binom{p_k+m_k-1}{m_k-1}I. \tag{3.2}$$

According to relations (2.1) and (3.1), for each $i\in\{1,\ldots,k\}$, and $p_i\in\mathbb{N}$, the operators $\{W_{i,\beta_i}\}_{\beta_i\in\mathbb{F}_{n_i},|\beta_i|=p_i}$ have orthogonal ranges and

$$\|W_{i,\beta_i}x\|\leq\frac{1}{\sqrt{b_{i,\beta_i}^{(m_i)}}}\binom{|\beta_i|+m_i-1}{m_i-1}^{1/2}\|x\|,\qquad x\in F^2(H_{n_i}).$$

Consequently, we deduce that

$$\sum_{\beta_i\in\mathbb{F}_{n_i}^+,|\beta_i|=p_i} b_{i,\beta_i}^{(m_i)}W_{i,\beta_i}W_{i,\beta_i}^*\leq\binom{p_i+m_i-1}{m_i-1}I\quad\text{for any}\quad p_i\in\mathbb{N}.$$

A repeated application of this inequality proves our assertion. Since $\varphi(\mathbf{W}_{i,j}) \in F^\infty(\mathbf{D_f^m})$, we have

$$\sum_{\beta_1\in\mathbb{F}_{n_1}^+,\ldots,\beta_k\in\mathbb{F}_{n_k}^+} |c_{\beta_1,\ldots,\beta_k}|^2 \frac{1}{b_{1,\beta_1}^{(m_1)}\cdots b_{k,\beta_k}^{(m_k)}} < \infty. \tag{3.3}$$

Hence, using relation (3.2) and Cauchy-Schwarz inequality, we deduce that, for $0 \le r < 1$,

$$\begin{aligned}
&\sum_{p=0}^\infty r^p \left\| \sum_{\substack{p_1,\ldots,p_k\in\mathbb{N}\cup\{0\}\\ p_1+\cdots+p_k=p}} \sum_{\substack{\beta_1\in\mathbb{F}_{n_1}^+,\ldots,\beta_k\in\mathbb{F}_{n_k}^+\\ |\beta_1|=p_1,\ldots,|\beta_k|=p_k}} c_{\beta_1,\ldots,\beta_k}\mathbf{W}_{1,\beta_1}\cdots\mathbf{W}_{k,\beta_k}\right\| \\
&\le \sum_{p=0}^\infty r^p \sum_{\substack{p_1,\ldots,p_k\in\mathbb{N}\cup\{0\}\\ p_1+\cdots+p_k=p}} \left(\sum_{\substack{\beta_1\in\mathbb{F}_{n_1}^+,\ldots,\beta_k\in\mathbb{F}_{n_k}^+\\ |\beta_1|=p_1,\ldots,|\beta_k|=p_k}} |c_{\beta_1,\ldots,\beta_k}|^2 \frac{1}{b_{1,\beta_1}^{(m_1)}\cdots b_{k,\beta_k}^{(m_k)}}\right)^{1/2} \\
&\qquad \left\| \sum_{\substack{\beta_1\in\mathbb{F}_{n_1}^+,\ldots,\beta_k\in\mathbb{F}_{n_k}^+\\ |\beta_1|=p_1,\ldots,|\beta_k|=p_k}} b_{1,\beta_1}^{(m_1)}\cdots b_{k,\beta_k}^{(m_k)}\mathbf{W}_{1,\beta_1}\cdots\mathbf{W}_{k,\beta_k}\mathbf{W}_{k,\beta_k}^*\cdots\mathbf{W}_{1,\beta_1}^*\right\|^{1/2} \\
&\le \sum_{p=0}^\infty r^p \sum_{\substack{p_1,\ldots,p_k\in\mathbb{N}\cup\{0\}\\ p_1+\cdots+p_k=p}} \binom{p_1+m_1-1}{m_1-1}^{1/2}\cdots\binom{p_k+m_k-1}{m_k-1}^{1/2} \left(\sum_{\substack{\beta_1\in\mathbb{F}_{n_1}^+,\ldots,\beta_k\in\mathbb{F}_{n_k}^+\\ |\beta_1|=p_1,\ldots,|\beta_k|=p_k}} |c_{\beta_1,\ldots,\beta_k}|^2 \frac{1}{b_{1,\beta_1}^{(m_1)}\cdots b_{k,\beta_k}^{(m_k)}}\right)^{1/2} \\
&\le \left(\sum_{p=0}^\infty r^{2p} \sum_{\substack{p_1,\ldots,p_k\in\mathbb{N}\cup\{0\}\\ p_1+\cdots+p_k=p}} \binom{p_1+m_1-1}{m_1-1}\cdots\binom{p_k+m_k-1}{m_k-1}\right)^{1/2} \left(\sum_{\beta_1\in\mathbb{F}_{n_1}^+,\ldots,\beta_k\in\mathbb{F}_{n_k}^+} |c_{\beta_1,\ldots,\beta_k}|^2 \frac{1}{b_{1,\beta_1}^{(m_1)}\cdots b_{k,\beta_k}^{(m_k)}}\right)^{1/2}.
\end{aligned}$$

Now, using relation (3.3) we obtain

$$\sum_{p=0}^\infty r^{2p} \sum_{\substack{p_1,\ldots,p_k\in\mathbb{N}\cup\{0\}\\ p_1+\cdots+p_k=p}} \binom{p_1+m_1-1}{m_1-1}\cdots\binom{p_k+m_k-1}{m_k-1} \le \sum_{p=0}^\infty r^{2p}(p+M)^{Mk-k}(p+1)^k < \infty, \tag{3.4}$$

where $M := \max\{m_1,\ldots,m_k\}$, and deduce that the series

$$\varphi(r\mathbf{W}_{i,j}) := \sum_{q=0}^\infty \sum_{\substack{\beta_1\in\mathbb{F}_{n_1}^+,\ldots,\beta_k\in\mathbb{F}_{n_k}^+\\ |\beta_1|+\cdots+|\beta_k|=q}} r^q c_{\beta_1,\ldots,\beta_k}\mathbf{W}_{1,\beta_1}\cdots\mathbf{W}_{k,\beta_k}$$

converges in the operator norm topology. Therefore $\varphi(r\mathbf{W}_{i,j})$ is in the noncommutative domain algebra $\mathcal{A}(\mathbf{D_f^m})$. In what follows, we show that

$$\varphi(\mathbf{W}_{i,j}) = \text{SOT-}\lim_{r\to1}\varphi(r\mathbf{W}_{i,j}) \tag{3.5}$$

for any $\varphi(\mathbf{W}_{i,j}) = \sum_{\beta_1\in\mathbb{F}_{n_1}^+,\ldots,\beta_k\in\mathbb{F}_{n_k}^+} c_{\beta_1,\ldots,\beta_k}\mathbf{W}_{1,\beta_1}\cdots\mathbf{W}_{k,\beta_k}$ in the noncommutative Hardy algebra $F^\infty(\mathbf{D_f^m})$. According to the first part of this lemma,

$$\varphi(r\mathbf{W}_{i,j}) = \lim_{n\to\infty} p_n(r\mathbf{W}_{i,j}), \tag{3.6}$$

where $p_n(\mathbf{W}_{i,j}) := \sum_{q=0}^n \sum_{\substack{\beta_1\in\mathbb{F}_{n_1}^+,\ldots,\beta_k\in\mathbb{F}_{n_k}^+\\ |\beta_1|+\cdots+|\beta_k|=q}} c_{\beta_1,\ldots,\beta_k}\mathbf{W}_{1,\beta_1}\cdots\mathbf{W}_{k,\beta_k}$ and the convergence is in the operator norm topology. For each $i\in\{1,\ldots,k\}$, let $\gamma_i,\sigma_i,\epsilon_i\in\mathbb{F}_{n_i}^+$ and set $n := |\gamma_1|+\cdots+|\gamma_k|$. Since $\mathbf{W}_{1,\beta_1}^*\cdots\mathbf{W}_{k,\beta_k}^*(e^1_{\gamma_1}\otimes\cdots\otimes e^k_{\gamma_k}) = 0$ for any $\beta_i\in\mathbb{F}_{n_i}^+$ with $|\beta_1|+\cdots+|\beta_k| > n$, we have

$$\varphi(r\mathbf{W}_{i,j})^*(e^1_{\alpha_1}\otimes\cdots\otimes e^k_{\alpha_k}) = p_n(r\mathbf{W}_{i,j})^*(e^1_{\alpha_1}\otimes\cdots\otimes e^k_{\alpha_k})$$

for any $\alpha_i \in \mathbb{F}_{n_i}^+$ with $|\alpha_1| + \cdots + |\alpha_k| \leq n$ and any $r \in [0,1)$. Due to Lemma 2.1 and Theorem 1.4, $r\mathbf{W} := (r\mathbf{W}_1, \ldots, r\mathbf{W}_n)$ is a pure k-tuple in the noncommutative polydomain $\mathbf{D}_{\mathbf{f}}^{\mathbf{m}}(\otimes_{i=1}^k F^2(H_{n_i}))$ for any $r \in [0,1)$. Applying Theorem 2.2, we obtain

$$\mathbf{K}_{\mathbf{f},r\mathbf{W}} p_n(r\mathbf{W}_{i,j})^* = [p_n(\mathbf{W}_{i,j})^* \otimes I_{\otimes_{i=1}^k F^2(H_{n_i})}]\mathbf{K}_{\mathbf{f},r\mathbf{W}}$$

for any $r \in [0,1)$. Using all these facts and the definition of the noncommutative Berezin kernel, careful calculations reveal that

$$\begin{aligned}
&\left\langle \mathbf{K}_{\mathbf{f},r\mathbf{W}} \varphi(r\mathbf{W}_{i,j})^*(e^1_{\gamma_1} \otimes \cdots \otimes e^k_{\gamma_k}), (e^1_{\sigma_1} \otimes \cdots \otimes e^k_{\sigma_k}) \otimes (e^1_{\epsilon_1} \otimes \cdots \otimes e^k_{\epsilon_k})\right\rangle \\
&= \left\langle \mathbf{K}_{\mathbf{f},r\mathbf{W}} p_n(r\mathbf{W}_{i,j})^*(e^1_{\gamma_1} \otimes \cdots \otimes e^k_{\gamma_k}), (e^1_{\sigma_1} \otimes \cdots \otimes e^k_{\sigma_k}) \otimes (e^1_{\epsilon_1} \otimes \cdots \otimes e^k_{\epsilon_k})\right\rangle \\
&= \left\langle [(p_n(\mathbf{W}_{i,j})^* \otimes I_{\otimes_{i=1}^k F^2(H_{n_i})})]\mathbf{K}_{\mathbf{f},r\mathbf{W}}(e^1_{\gamma_1} \otimes \cdots \otimes e^k_{\gamma_k}), (e^1_{\sigma_1} \otimes \cdots \otimes e^k_{\sigma_k}) \otimes (e^1_{\epsilon_1} \otimes \cdots \otimes e^k_{\epsilon_k})\right\rangle \\
&= \sum_{\beta_i \in \mathbb{F}_{n_i}^+, i=1,\ldots,k} r^{|\beta_1|+\cdots|\beta_k|} \sqrt{b^{(m_1)}_{1,\beta_1}} \cdots \sqrt{b^{(m_k)}_{k,\beta_k}} \left\langle p_n(\mathbf{W}_{i,j})^*(e^1_{\beta_1} \otimes \cdots \otimes e^k_{\beta_k}), e^1_{\sigma_1} \otimes \cdots \otimes e^k_{\sigma_k}\right\rangle \\
&\qquad \times \left\langle \mathbf{W}^*_{1,\beta_1} \cdots \mathbf{W}^*_{k,\beta_k}(e^1_{\gamma_1} \otimes \cdots \otimes e^k_{\gamma_k}), \boldsymbol{\Delta}^{\mathbf{m}}_{\mathbf{f},r\mathbf{W}}(I)^{1/2}(e^1_{\epsilon_1} \otimes \cdots \otimes e^k_{\epsilon_k})\right\rangle \\
&= \sum_{\beta_i \in \mathbb{F}_{n_i}^+, i=1,\ldots,k} r^{|\beta_1|+\cdots|\beta_k|} \sqrt{b^{(m_1)}_{1,\beta_1}} \cdots \sqrt{b^{(m_k)}_{k,\beta_k}} \left\langle \varphi(\mathbf{W}_{i,j})^*(e^1_{\beta_1} \otimes \cdots \otimes e^k_{\beta_k}), e^1_{\sigma_1} \otimes \cdots \otimes e^k_{\sigma_k}\right\rangle \\
&\qquad \times \left\langle \mathbf{W}^*_{1,\beta_1} \cdots \mathbf{W}^*_{k,\beta_k}(e^1_{\gamma_1} \otimes \cdots \otimes e^k_{\gamma_k}), \boldsymbol{\Delta}^{\mathbf{m}}_{\mathbf{f},r\mathbf{W}}(I)^{1/2}(e^1_{\epsilon_1} \otimes \cdots \otimes e^k_{\epsilon_k})\right\rangle \\
&= \left\langle [(\varphi(\mathbf{W}_{i,j})^* \otimes I_{\otimes_{i=1}^k F^2(H_{n_i})})]\mathbf{K}_{\mathbf{f},r\mathbf{W}}(e^1_{\gamma_1} \otimes \cdots \otimes e^k_{\gamma_k}), (e^1_{\sigma_1} \otimes \cdots \otimes e^k_{\sigma_k}) \otimes (e^1_{\epsilon_1} \otimes \cdots \otimes e^k_{\epsilon_k})\right\rangle
\end{aligned}$$

for any $r \in [0,1)$ and $\gamma_i, \sigma_i, \epsilon_i \in \mathbb{F}_{n_i}^+$, $i \in \{1, \ldots, k\}$. Hence, since $\varphi(r\mathbf{W}_{i,j})$ and $\varphi(\mathbf{W}_{i,j})$ are bounded operators on $\otimes_{i=1}^k F^2(H_{n_i})$, we deduce that

$$\mathbf{K}_{\mathbf{f},r\mathbf{W}} \varphi(r\mathbf{W}_{i,j})^* = [\varphi(\mathbf{W}_{i,j})^* \otimes I_{\otimes_{i=1}^k F^2(H_{n_i})}]\mathbf{K}_{\mathbf{f},r\mathbf{W}}$$

for any $r \in [0,1)$. Since $r\mathbf{W} := (r\mathbf{W}_1, \ldots, r\mathbf{W}_n)$ is a pure k-tuple in the noncommutative polydomain $\mathbf{D}_{\mathbf{f}}^{\mathbf{m}}(\otimes_{i=1}^k F^2(H_{n_i}))$ for any $r \in [0,1)$, Theorem 2.2 shows that the Berezin kernel $\mathbf{K}_{\mathbf{f},r\mathbf{W}}$ is an isometry and, therefore, the equality above implies

$$\|\varphi(r\mathbf{W}_{i,j}))\| \leq \|\varphi(\mathbf{W}_{i,j}))\| \quad \text{for any } r \in [0,1). \tag{3.7}$$

Hence, and due to the fact that $\varphi(\mathbf{W}_{i,j})(e^1_{\alpha_1} \otimes \cdots \otimes e^k_{\alpha_k}) = \lim_{r\to 1} \varphi(r\mathbf{W}_{i,j})(e^1_{\alpha_1} \otimes \cdots \otimes e^k_{\alpha_k})$ for any $\alpha_i \in \mathbb{F}_{n_i}^+$, an approximation argument implies relation (3.5). Note that if $0 < r_1 < r_2 < 1$, then

$$\|\varphi(r_1\mathbf{W}_{i,j})\| \leq \|\varphi(r_2\mathbf{W}_{i,j})\|.$$

Indeed, since $\varphi(r_2\mathbf{W}_{i,j})$ is in the polydomain algebra $\mathcal{A}(\mathbf{D}_{\mathbf{f}}^{\mathbf{m}})$, Theorem 2.4 implies $\|\varphi(rr_2\mathbf{W}_{i,j})\| \leq \|\varphi(r_2\mathbf{W}_{i,j})\|$ for any $r \in [0,1)$. Taking $r := \frac{r_1}{r_2}$, we prove our assertion. Now one can easily complete the proof of the theorem. ☐

Lemma 3.4. *Let $\mathbf{T} = (T_1, \ldots, T_k)$ be in the noncommutative polydomain $\mathbf{D}_{\mathbf{f}}^{\mathbf{m}}(\mathcal{H})$ and let $\varphi(\mathbf{W}_{i,j})$ be in the Hardy algebra $F^\infty(\mathbf{D}_{\mathbf{f}}^{\mathbf{m}})$. Then the noncommutative Berezin kernel satisfies the relations*

$$\varphi(rT_{i,j})\mathbf{K}^*_{\mathbf{f},\mathbf{T}} = \mathbf{K}^*_{\mathbf{f},\mathbf{T}}(\varphi(r\mathbf{W}_{i,j}) \otimes I_{\mathcal{H}})$$

and

$$\varphi(rT_{i,j})\mathbf{K}^*_{\mathbf{f},r\mathbf{T}} = \mathbf{K}^*_{\mathbf{f},r\mathbf{T}}(\varphi(\mathbf{W}_{i,j}) \otimes I_{\mathcal{H}})$$

for any $r \in [0,1)$.

Proof. Due to Theorem 2.2, we have

$$T_{i,j}\mathbf{K}^*_{\mathbf{f},\mathbf{T}} = \mathbf{K}^*_{\mathbf{f},\mathbf{T}}(\mathbf{W}_{i,j} \otimes I_{\mathcal{H}})$$

for any $i \in \{1, \dots, k\}$ and $j \in \{1, \dots, n_i\}$. Hence, using Theorem 2.4 and part (i) of Lemma 3.3, we deduce that

$$\varphi(rT_{i,j}) := \sum_{q=0}^{\infty} \sum_{\substack{\beta_1 \in \mathbb{F}_{n_1}^+, \dots, \beta_k \in \mathbb{F}_{n_k}^+ \\ |\beta_1| + \dots + |\beta_k| = q}} r^q c_{\beta_1, \dots, \beta_k} T_{1,\beta_1} \cdots T_{k,\beta_k}$$

converges in the operator norm topology and $\varphi(rT_{i,j})\mathbf{K}_{\mathbf{f},\mathbf{T}}^* = \mathbf{K}_{\mathbf{f},\mathbf{T}}^*(\varphi(r\mathbf{W}_{i,j}) \otimes I_{\mathcal{H}})$ for all $r \in [0,1)$. Now, we prove the second part of this lemma. Using again Theorem 2.2, we obtain

$$\mathbf{K}_{\mathbf{f},\mathbf{rT}}^*[p(\mathbf{W}_{i,j}) \otimes I_{\mathcal{H}}] = p(rT_{i,j})\mathbf{K}_{\mathbf{f},\mathbf{rT}}^* \tag{3.8}$$

for any polynomial $p(\mathbf{W}_{i,j})$ and $r \in [0,1)$. Since $r\mathbf{T} := (rT_1, \dots, rT_n) \in \mathbf{D}_{\mathbf{f}}^{\mathbf{m}}(\mathcal{H})$ (see Theorem 1.4), relation (3.6) and Theorem 2.4 imply

$$\varphi(rtT_{i,j}) = \lim_{n \to \infty} \sum_{q=0}^{n} \sum_{\substack{\beta_1 \in \mathbb{F}_{n_1}^+, \dots, \beta_k \in \mathbb{F}_{n_k}^+ \\ |\beta_1| + \dots + |\beta_k| = q}} (rt)^q c_{\beta_1, \dots, \beta_k} T_{1,\beta_1} \cdots T_{k,\beta_k}, \qquad r, t \in [0,1),$$

where the convergence is in the operator norm topology. Consequently, an approximation argument shows that relation (3.8) implies

$$\mathbf{K}_{\mathbf{f},\mathbf{rT}}^*[\varphi(t\mathbf{W}_{i,j}) \otimes I_{\mathcal{H}}] = \varphi(rtT_{i,j})\mathbf{K}_{\mathbf{f},\mathbf{rT}}^* \quad \text{for} \ \ r, t \in (0,1). \tag{3.9}$$

On the other hand, let us prove that

$$\lim_{t \to 1} \varphi(rtT_{i,j}) = \varphi(rT_{i,j}), \tag{3.10}$$

where the convergence is in the operator norm topology. Notice that, due to relation (3.4), if $\epsilon > 0$, there is $m_0 \in \mathbb{N}$ such that $\sum_{p=m_0}^{\infty} r^{2p} \sum_{\substack{p_1, \dots, p_k \in \mathbb{N} \cup \{0\} \\ p_1 + \dots + p_k = p}} \binom{p_1 + m_1 - 1}{m_1 - 1} \cdots \binom{p_k + m_k - 1}{m_k - 1} < \frac{\epsilon^2}{4K^2}$, where $K := \|\varphi(\mathbf{W}_{i,j})(1)\|$. Since $\mathbf{T} := (T_1, \dots, T_n) \in \mathbf{D}_{\mathbf{f}}^{\mathbf{m}}(\mathcal{H})$, Theorem 2.4 and relation (3.2) imply

$$\begin{aligned} &\sum_{\substack{\beta_1 \in \mathbb{F}_{n_1}^+, \dots, \beta_k \in \mathbb{F}_{n_k}^+ \\ |\beta_1| = p_1, \dots, |\beta_k| = p_k}} b_{1,\beta_1}^{(m_1)} \cdots b_{k,\beta_k}^{(m_k)} T_{1,\beta_1} \cdots T_{k,\beta_k} T_{k,\beta_k}^* \cdots T_{1,\beta_1}^* \\ &\qquad\qquad \le \binom{p_1 + m_1 - 1}{m_1 - 1} \cdots \binom{p_k + m_k - 1}{m_k - 1} I. \end{aligned}$$

Now, as in the proof of Lemma 3.3, we can deduce that

$$\begin{aligned} &\sum_{p=m_0}^{\infty} r^p \left\| \sum_{\substack{p_1, \dots, p_k \in \mathbb{N} \cup \{0\} \\ p_1 + \dots + p_k = p}} \sum_{\substack{\beta_1 \in \mathbb{F}_{n_1}^+, \dots, \beta_k \in \mathbb{F}_{n_k}^+ \\ |\beta_1| = p_1, \dots, |\beta_k| = p_k}} c_{\beta_1, \dots, \beta_k} T_{1,\beta_1} \cdots T_{k,\beta_k} \right\| \\ &\le \left(\sum_{p=m_0}^{\infty} r^{2p} \sum_{\substack{p_1, \dots, p_k \in \mathbb{N} \cup \{0\} \\ p_1 + \dots + p_k = p}} \binom{p_1 + m_1 - 1}{m_1 - 1} \cdots \binom{p_k + m_k - 1}{m_k - 1} \right)^{1/2} \|\varphi(\mathbf{W}_{i,j})(1)\| \\ &\le \frac{\epsilon}{2}. \end{aligned}$$

Consequently, setting $T_{(\beta)} := T_{1,\beta_1}\cdots T_{k,\beta_k}$, there exists $0<d<1$ such that

$$\begin{aligned}&\left\|\sum_{p=0}^\infty \sum_{\substack{\beta_1\in\mathbb{F}_{n_1}^+,\ldots,\beta_k\in\mathbb{F}_{n_k}^+\\ |\beta_1|+\cdots+|\beta_k|=p}} (rt)^{|\beta_1|+\cdots+|\beta_k|}c_{\beta_1,\ldots,\beta_k}T_{(\beta)} - \sum_{p=0}^\infty \sum_{\substack{\beta_1\in\mathbb{F}_{n_1}^+,\ldots,\beta_k\in\mathbb{F}_{n_k}^+\\ |\beta_1|+\cdots+|\beta_k|=p}} r^{|\beta_1|+\cdots+|\beta_k|}c_{\beta_1,\ldots,\beta_k}T_{(\beta)}\right\|\\
&\leq \epsilon + \left\|\sum_{p=1}^{m_0-1} r^p(t^p-1)\sum_{\substack{\beta_1\in\mathbb{F}_{n_1}^+,\ldots,\beta_k\in\mathbb{F}_{n_k}^+\\ |\beta_1|+\cdots+|\beta_k|=p}} c_{\beta_1,\ldots,\beta_k}T_{(\beta)}\right\| \|\varphi(\mathbf{W}_{i,j})(1)\|\\
&\leq 2\epsilon\end{aligned}$$

for any $t\in(d,1)$. Hence, we deduce relation (3.10). On the other hand, due to Lemma 3.3, we have $\varphi(\mathbf{W}_{i,j}) = \text{SOT-}\lim_{t\to 1}\varphi(t\mathbf{W}_{i,j})$. Since the map $Y\mapsto Y\otimes I_\mathcal{H}$ is SOT-continuous on bounded sets, we deduce that

$$\text{SOT-}\lim_{t\to 1}[\varphi(t\mathbf{W}_{i,j}))\otimes I_\mathcal{H}] = \varphi(\mathbf{W}_{i,j})\otimes I_\mathcal{H}. \tag{3.11}$$

Consequently, using relation (3.10) and passing to limit in (3.9), as $t\to 1$, we complete the proof. $\square$

In what follows we show that the restriction of the noncommutative Berezin transform to the Hardy algebra $F^\infty(\mathbf{D_f^m})$ provides a functional calculus associated with each pure tuple of operators in the noncommutative domain $\mathbf{D_f^m}(\mathcal{H})$. Moreover, we obtain a Fatou type result.

Theorem 3.5. *Let $\mathbf{T}=(T_1,\ldots,T_k)$ be a pure k-tuple in the noncommutative polydomain $\mathbf{D_f^m}(\mathcal{H})$ and define the map*

$$\Psi_\mathbf{T} : F^\infty(\mathbf{D_f^m})\to B(\mathcal{H}) \quad \text{by} \quad \Psi_\mathbf{T}(\varphi) := \mathbf{B_T}[\varphi],$$

where $\mathbf{B_T}$ is the noncommutative Berezin transform at $\mathbf{T}\in\mathbf{D_f^m}(\mathcal{H})$. Then

(i) *$\Psi_\mathbf{T}$ is WOT-continuous (resp. SOT-continuous) on bounded sets;*

(ii) *$\Psi_\mathbf{T}$ is a unital completely contractive homomorphism and*

$$\Psi_\mathbf{T}(\mathbf{W}_{1,\beta_1}\cdots\mathbf{W}_{k,\beta_k}) = T_{1,\beta_1}\cdots T_{k,\beta_k}, \qquad \beta_i\in\mathbb{F}_{n_i}^+, i\in\{1,\ldots,k\}$$

(iii) *for any $\varphi(\mathbf{W}_{i,j})\in F^\infty(\mathbf{D_f^m})$,*

$$\mathbf{B}_{r\mathbf{T}}[\varphi(\mathbf{W}_{i,j})] = \varphi(rT_{i,j}) = \mathbf{B_T}[\varphi(r\mathbf{W}_{i,j})]$$

and

$$\Psi_\mathbf{T}(\varphi(\mathbf{W}_{i,j})) = \text{SOT-}\lim_{r\to 1}\varphi(rT_{i,j}).$$

Proof. Since

$$\Psi_\mathbf{T}(\varphi(\mathbf{W}_{i,j})) = \mathbf{K}_{\mathbf{f},\mathbf{T}}^*(\varphi(\mathbf{W}_{i,j})\otimes I_\mathcal{H})\mathbf{K}_{\mathbf{f},\mathbf{T}}, \qquad \varphi(\mathbf{W}_{i,j})\in F^\infty(\mathbf{D}_f^m), \tag{3.12}$$

using standard facts in functional analysis, we deduce part (i).

Now, we prove part (ii). Since $\mathbf{T}$ is pure, Theorem 2.2 shows that $\mathbf{K}_{\mathbf{f},\mathbf{T}}$ is an isometry. Consequently, relation (3.12) implies

$$\left\|[\Psi_\mathbf{T}(\varphi_{ij})]_{k\times k}\right\| \leq \left\|[\varphi_{ij}]_{k\times k}\right\|$$

for any operator-valued matrix $[\varphi_{ij}]_{k\times k}$ in $M_{k\times k}(F^\infty(\mathbf{D}_f^m))$, which proves that $\Psi_\mathbf{T}$ is a unital completely contractive linear map. Due to Theorem 2.4, $\Psi_\mathbf{T}$ is a homomorphism on the set $\mathcal{P}(\mathbf{W})$ of polynomials in $\{\mathbf{W}_{i,j}\}$. By Proposition 3.2, the polynomials in $\mathbf{W}_{i,j}$ and the identity are sequentially WOT-dense in $F^\infty(\mathbf{D}_f^m)$. On the other hand, due to part (i), Ψ_T is WOT- continuous on bounded sets. Using the principle of uniform boundedness we deduce that Ψ_T is also a homomorphism on $F^\infty(\mathbf{D}_f^m)$.

Due to Lemma 3.4 and taking into account that $\mathbf{K}_{\mathbf{f},\mathbf{T}}$ and $\mathbf{K}_{\mathbf{f},\mathbf{rT}}$ are isometries, we have

$$\begin{aligned}\mathbf{B}_{r\mathbf{T}}[\varphi(\mathbf{W}_{i,j})] &= \mathbf{K}_{\mathbf{f},\mathbf{rT}}^*(\varphi(\mathbf{W}_{i,j})\otimes I)\mathbf{K}_{\mathbf{f},\mathbf{rT}}\\ &= \varphi(rT_{i,j}) = \mathbf{K}_{\mathbf{f},\mathbf{T}}^*(\varphi(r\mathbf{W}_{i,j})\otimes I)\mathbf{K}_{\mathbf{f},\mathbf{T}}\\ &= \mathbf{B_T}[\varphi(r\mathbf{W}_{i,j})].\end{aligned}$$

Now, due to relation (3.11) we have

$$\text{SOT-}\lim_{t\to 1}[\varphi(t\mathbf{W}_{i,j}))\otimes I_{\mathcal{H}}] = \varphi(\mathbf{W}_{i,j})\otimes I_{\mathcal{H}}.$$

Hence, and using the equalities above, we deduce that

$$\begin{aligned}\mathbf{B}_{\mathbf{T}}[\varphi(\mathbf{W}_{i,j})] &:= \mathbf{K}_{\mathbf{f},\mathbf{T}}^*(\varphi(\mathbf{W}_{i,j})\otimes I)\mathbf{K}_{\mathbf{f},\mathbf{T}}\\ &= \text{SOT-}\lim_{r\to 1}\mathbf{K}_{\mathbf{f},\mathbf{T}}^*(\varphi(r\mathbf{W}_{i,j})\otimes I)\mathbf{K}_{\mathbf{f},\mathbf{T}}\\ &= \text{SOT-}\lim_{r\to 1}\varphi(rT_{i,j}).\end{aligned}$$

This completes the proof. □

We say that $\mathbf{T} = (T_1,\ldots,T_k)\in \mathbf{D}_{\mathbf{f}}^{\mathbf{m}}(\mathcal{H})$ is completely non-coisometric if there is no $h\in\mathcal{H}$, $h\neq 0$ such that

$$\left\langle (id-\Phi_{f_1,T_1}^{q_1})\cdots(id-\Phi_{f_k,T_k}^{q_k})(I_{\mathcal{H}})h,h\right\rangle = 0$$

for any $(q_1,\ldots,q_k)\in\mathbb{N}^k$. This is equivalent to the condition

$$\lim_{q=(q_1,\ldots,q_k)\in\mathbb{N}^k}\left\langle (id-\Phi_{f_1,T_1}^{q_1})\cdots(id-\Phi_{f_k,T_k}^{q_k})(I_{\mathcal{H}})h,h\right\rangle = 0.$$

In what follows we present an $F^\infty(\mathbf{D}_{\mathbf{f}}^{\mathbf{m}})$-functional calculus for the completely non-coisometric part of the noncommutative polydomain $\mathbf{D}_{\mathbf{f}}^{\mathbf{m}}(\mathcal{H})$.

Theorem 3.6. *Let $\mathbf{T} = (T_1,\ldots,T_k)$ be a completely non-coisometric k-tuple in the noncommutative polydomain $\mathbf{D}_{\mathbf{f}}^{\mathbf{m}}(\mathcal{H})$. Then*

$$\Phi(\varphi) := \text{SOT-}\lim_{r\to 1}\varphi(rT_{i,j}),\qquad \varphi=\varphi(\mathbf{W}_{i,j})\in F^\infty(\mathbf{D}_{\mathbf{f}}^{\mathbf{m}}),$$

exists in the strong operator topology and defines a map $\Phi: F^\infty(\mathbf{D}_{\mathbf{f}}^{\mathbf{m}})\to B(\mathcal{H})$ with the following properties:

(i) $\Phi(\varphi) = \text{SOT-}\lim_{r\to 1}\mathbf{B}_{r\mathbf{T}}[\varphi]$, *where $\mathbf{B}_{r\mathbf{T}}$ is the noncommutative Berezin transform at $r\mathbf{T}\in\mathbf{D}_{\mathbf{f}}^{\mathbf{m}}(\mathcal{H})$;*
(ii) *Φ is WOT-continuous (resp. SOT-continuous) on bounded sets;*
(iii) *Φ is a unital completely contractive homomorphism.*

Proof. According to Theorem 1.4, $r\mathbf{T}\in\mathbf{D}_{\mathbf{f}}^{\mathbf{m}}(\mathcal{H})$ and $r\mathbf{W}\in\mathbf{D}_{\mathbf{f}}^{\mathbf{m}}(\otimes_{i=1}^k F^2(H_{n_i}))$ for any $r\in[0,1)$. Due to relations (3.7) and (3.11), we have $\text{SOT-}\lim_{t\to 1}[\varphi(t\mathbf{W}_{i,j}))\otimes I_{\mathcal{H}}] = \varphi(\mathbf{W}_{i,j})\otimes I_{\mathcal{H}}$. Taking the limit in the first relation of Lemma 3.4, as $r\to 1$, we deduce that the map $\Lambda:\operatorname{range}\mathbf{K}_{\mathbf{f},\mathbf{T}}^*\to\mathcal{H}$ given by $\Lambda y := \lim_{r\to 1}\varphi(tT_{i,j})y$, $y\in\operatorname{range}\mathbf{K}_{\mathbf{f},\mathbf{T}}^*$, is well-defined, linear, and

$$\|\Lambda\mathbf{K}_{\mathbf{f},\mathbf{T}}^*x\|\leq\limsup_{r\to 1}\|\varphi(rT_{i,j})\|\|\mathbf{K}_{\mathbf{f},\mathbf{T}}^*x\|\leq\|\varphi(\mathbf{W}_{i,j})\|\|\mathbf{K}_{\mathbf{f},\mathbf{T}}^*x\|$$

for any $x\in(\otimes_{i=1}^k F^2(H_{n_i}))\otimes\mathcal{H}$.

Since $\mathbf{T}=(T_1,\ldots,T_k)$ is completely non-coisometric, Theorem 2.2 implies that the noncommutative Berezin kernel $\mathbf{K}_{\mathbf{f},\mathbf{T}}$ is one-to-one and, therefore, the range of $\mathbf{K}_{\mathbf{f},\mathbf{T}}^*$ is dense in $\mathcal{H}$. Consequently, the map Λ has a unique extension to a bounded linear operator on $\mathcal{H}$, denoted also by Λ, with $\|\Lambda\|\leq\|\varphi(\mathbf{W}_{i,j})\|$. We show that

$$\lim_{r\to 1}\varphi(rT_{i,j})h = \Lambda h\quad\text{ for any }\ h\in\mathcal{H}. \tag{3.13}$$

Let $h\in\mathcal{H}$ and let $\{y_k\}_{k=1}^\infty$ be a sequence of vectors in the range of $\mathbf{K}_{\mathbf{f},\mathbf{T}}^*$, which converges to h. According to Theorem 2.4 and relations (3.6), (3.7), we have

$$\|\varphi(rT_{i,j})\|\leq\|\varphi(r\mathbf{W}_{i,j})\|\leq\|\varphi(\mathbf{W}_{i,j})\|$$

for any $r\in[0,1)$. Note that

$$\begin{aligned}\|\Lambda h-\varphi(rT_{i,j}))h\| &\leq \|\Lambda h-\Lambda y_k\| + \|\Lambda y_k-\varphi(rT_{i,j}))y_k\| + \|\varphi(rT_{i,j}))y_k-\varphi(rT_{i,j}))h\|\\ &\leq 2\|\varphi(\mathbf{W}_{i,j})\|\|h-y_k\| + \|\Lambda y_k-\varphi(rT_{i,j}))y_k\|.\end{aligned}$$

Consequently, since $\lim_{r\to 1}\varphi(rT_{i,j})y_k = \Lambda y_k$, relation (3.13) follows. Due to Lemma 3.4, we have

$$\varphi(rT_{i,j}) = \mathbf{K}^*_{\mathbf{f},\mathbf{rT}}[\varphi(\mathbf{W}_{i,j})\otimes I_{\mathcal{H}}]\mathbf{K}_{\mathbf{f},\mathbf{rT}}, \tag{3.14}$$

which together with relation (3.13) imply part (i) of the theorem.

Now we prove part (ii). Since $\|\varphi(rT_{i,j})\| \leq \|\varphi(\mathbf{W}_{i,j})\|$ we deduce that $\|\Phi(\varphi)\| \leq \|\varphi\|$ for $\varphi \in F^\infty(\mathbf{D}^{\mathbf{m}}_{\mathbf{f}})$. Taking $r\to 1$ in the first relation of Lemma 3.4 and using the first part of this theorem, we obtain

$$\Phi(\varphi)\mathbf{K}^*_{\mathbf{f},\mathbf{T}} = \mathbf{K}^*_{\mathbf{f},\mathbf{T}}(\varphi\otimes I), \quad \varphi \in F^\infty(\mathbf{D}^{\mathbf{m}}_{\mathbf{f}}). \tag{3.15}$$

If $\{g_\iota\}$ be a bounded net in $F^\infty(\mathbf{D}^{\mathbf{m}}_{\mathbf{f}})$ such that $g_\iota \to g \in F^\infty(\mathbf{D}^{\mathbf{m}}_{\mathbf{f}})$ in the weak (resp. strong) operator topology, then $g_\iota\otimes I$ converges to $g\otimes I$ in the same topologies. By relation (3.15), we have $\Phi(g_\iota)\mathbf{K}^*_{\mathbf{f},\mathbf{T}} = \mathbf{K}^*_{\mathbf{f},\mathbf{T}}(g_\iota\otimes I)$. Since the range of $\mathbf{K}^*_{\mathbf{f},\mathbf{T}}$ is dense in $\mathcal{H}$ and $\{\Phi(g_\iota)\}$ is bounded, an approximation argument shows that $\Phi(g_\iota)\to\Phi(g)$ in the weak (resp. strong) operator topology.

Now, we prove (iii). Relation (3.14) and the fact that $\mathbf{K}_{\mathbf{f},\mathbf{rT}}$ is an isometry for $r\in[0,1)$ imply

$$\|[\varphi_{st}(rT_{i,j})]_{k\times k}\| \leq \|[\varphi_{st}]_{k\times k}\|$$

for any operator-valued matrix $[\varphi_{st}]_{k\times k} \in M_{k\times k}(F^\infty(\mathbf{D}^{\mathbf{m}}_{\mathbf{f}}))$ and $r\in[0,1)$. Hence, and using the fact that $\Phi(\varphi_{st}) = \text{SOT-}\lim_{r\to 1}\varphi_{st}(rT_{i,j})$, we deduce that Φ is completely contractive map. Due to Theorem 2.4, Φ is a homomorphism on polynomials in $\mathbf{W}_{i,j}$ and the identity. Since, due to Proposition 3.2, these polynomials are sequentially WOT-dense in $F^\infty(\mathbf{D}^{\mathbf{m}}_{\mathbf{f}})$ and Φ is WOT-continuous on bounded sets, we deduce part (iii) of the theorem. The proof is complete. □

4. Free holomorphic functions on noncommutative polydomains

We introduce the algebra $Hol(\mathbf{D}^{\mathbf{m}}_{\mathbf{f},\text{rad}})$ of all free holomorphic functions on the abstract radial polydomain $\mathbf{D}^{\mathbf{m}}_{\mathbf{f},\text{rad}}$. We identify the polydomain algebra $\mathcal{A}(\mathbf{D}^{\mathbf{m}}_{\mathbf{f}})$ and the Hardy algebra $F^\infty(\mathbf{D}^{\mathbf{m}}_{\mathbf{f}})$ with subalgebras of $Hol(\mathbf{D}^{\mathbf{m}}_{\mathbf{f},\text{rad}})$.

For each $i\in\{1,\ldots,k\}$, let $Z_i := (Z_{i,1},\ldots,Z_{i,n_i})$ be an n_i-tuple of noncommuting indeterminates and assume that, for any $p,q\in\{1,\ldots,k\}$, $p\neq q$, the entries in Z_p are commuting with the entries in Z_q. We set $Z_{i,\alpha_i} := Z_{i,j_1}\cdots Z_{i,j_p}$ if $\alpha_i\in\mathbb{F}^+_{n_i}$ and $\alpha_i = g^i_{j_1}\cdots g^i_{j_p}$, and $Z_{i,g^i_0} := 1$, where g^i_0 is the identity in $\mathbb{F}^+_{n^i}$. We consider formal power series

$$\varphi = \sum_{\alpha_1\in\mathbb{F}^+_{n_1},\ldots,\alpha_k\in\mathbb{F}^+_{n_k}} a_{\alpha_1,\ldots,\alpha_k} Z_{1,\alpha_1}\cdots Z_{k,\alpha_k}, \qquad a_{\alpha_1,\ldots,\alpha_k}\in\mathbb{C},$$

in ideterminates $Z_{i,j}$. Denoting $(\alpha) := (\alpha_1,\ldots,\alpha_k) \in \mathbb{F}^+_{n_1}\times\cdots\times\mathbb{F}^+_{n_k}$, $Z_{(\alpha)} := Z_{1,\alpha_1}\cdots Z_{k,\alpha_k}$, and $a_{(\alpha)} := a_{\alpha_1,\ldots,\alpha_k}$, we can also use the abbreviation $\varphi = \sum_{(\alpha)} a_{(\alpha)}Z_{(\alpha)}$.

Given a Hilbert space $\mathcal{H}$, we introduce the radial polydomain

$$\mathbf{D}^{\mathbf{m}}_{\mathbf{f},\text{rad}}(\mathcal{H}) := \bigcup_{0\leq r<1} r\mathbf{D}^{\mathbf{m}}_{\mathbf{f}}(\mathcal{H}) \subseteq \mathbf{D}^{\mathbf{m}}_{\mathbf{f}}(\mathcal{H}).$$

A formal power series φ, having the representation above, is called free holomorphic function on the *abstract radial polydomain* $\mathbf{D}^{\mathbf{m}}_{\mathbf{f},\text{rad}} := \{\mathbf{D}^{\mathbf{m}}_{\mathbf{f},\text{rad}}(\mathcal{H}) :\ \mathcal{H} \text{ is a Hilbert space}\}$ if the series

$$\varphi(X_{i,j}) := \sum_{q=0}^\infty \sum_{\substack{(\alpha)\in\mathbb{F}^+_{n_1}\times\cdots\times\mathbb{F}^+_{n_k}\\ |\alpha_1|+\cdots+|\alpha_k|=q}} a_{(\alpha)}X_{(\alpha)}$$

is convergent in the operator norm topology for any $X = (X_{i,j}) \in \mathbf{D}^{\mathbf{m}}_{\mathbf{f},\text{rad}}(\mathcal{H})$ with $i\in\{1,\ldots,k\}$, $j\in\{1,\ldots,n_i\}$, and any Hilbert space $\mathcal{H}$. We denote by $Hol(\mathbf{D}^{\mathbf{m}}_{\mathbf{f},\text{rad}})$ the set of all free holomorphic functions on the abstract radial polydomain $\mathbf{D}^{\mathbf{m}}_{\mathbf{f},\text{rad}}$.

Lemma 4.1. *Let* $\varphi = \sum_{(\alpha)\in \mathbb{F}_{n_1}^+\times\cdots\times\mathbb{F}_{n_k}^+} a_{(\alpha)} Z_{(\alpha)}$ *be a formal power series and let* $\mathbf{W}=\{\mathbf{W}_{i,j}\}$ *be the universal model associated with the abstract noncommutative polydomain* $\mathbf{D}_{\mathbf{f}}^{\mathbf{m}}$*. Then the following statements are equivalent.*

(i) *φ is a free holomorphic function on the abstract radial polydomain* $\mathbf{D}_{\mathbf{f},\text{rad}}^{\mathbf{m}}$.

(ii) *For any $r\in[0,1)$, the series*

$$\varphi(r\mathbf{W}_{i,j}) := \sum_{q=0}^{\infty} \sum_{\substack{(\alpha)\in \mathbb{F}_{n_1}^+\times\cdots\times\mathbb{F}_{n_k}^+\\ |\alpha_1|+\cdots+|\alpha_k|=q}} a_{(\alpha)} r^{|\alpha_1|+\cdots+|\alpha_k|}\mathbf{W}_{(\alpha)}$$

is convergent in the operator norm topology.

(iii) *The inequality*

$$\limsup_{n\to\infty}\left\| \sum_{\substack{(\alpha)\in \mathbb{F}_{n_1}^+\times\cdots\times\mathbb{F}_{n_k}^+\\ |\alpha_1|+\cdots+|\alpha_k|=n}} a_{(\alpha)}\mathbf{W}_{(\alpha)}\right\|^{1/n}\leq 1.$$

Proof. The equivalence of (i) with (ii) is due to Theorem 2.4. Using standard arguments, one can easily prove that (ii) is equivalent to (iii). $\square$

We remark that the coefficients of a free holomorphic function are uniquely determined by its representation on an infinite dimensional Hilbert space. Indeed, under the above notations, let $0<r<1$ and assume that $\varphi(r\mathbf{W}_{i,j})=0$. Taking into account relation (2.6), we have

$$\left\langle \varphi(r\mathbf{W}_{i,j})1, \mathbf{W}_{(\alpha)}1\right\rangle = r^{|\alpha_1|+\cdots+|\alpha_k|} a_{(\alpha)} \frac{1}{b_{1,\alpha_1}^{(m_1)}}\cdots \frac{1}{b_{k,\alpha_k}^{(m_k)}} = 0$$

for any $(\alpha)=(\alpha_1,\ldots,\alpha_k)\in \mathbb{F}_{n_1}^+\otimes\cdots\otimes\mathbb{F}_{n_k}^+$. Therefore $a_{(\alpha)}=0$, which proves our assertion.

Due to Lemma 4.1, if $\varphi\in Hol(\mathbf{D}_{\mathbf{f},\text{rad}}^{\mathbf{m}})$, then $\varphi(r\mathbf{W}_{i,j})$ is in the domain algebra $\mathcal{A}(\mathbf{D}_{\mathbf{f}}^{\mathbf{m}})$ for any $r\in[0,1)$. Using the results from the previous section, one can see that $Hol(\mathbf{D}_{\mathbf{f},\text{rad}}^{\mathbf{m}})$ is an algebra. Let $H^\infty(\mathbf{D}_{\mathbf{f},\text{rad}}^{\mathbf{m}})$ denote the set of all elements φ in $Hol(\mathbf{D}_{\mathbf{f},\text{rad}}^{\mathbf{m}})$ such that

$$\|\varphi\|_\infty := \sup \|\varphi(X_{i,j})\|<\infty,$$

where the supremum is taken over all $(X_{i,j})\in \mathbf{D}_{\mathbf{f},\text{rad}}^{\mathbf{m}}(\mathcal{H})$ and any Hilbert space $\mathcal{H}$. One can show that $H^\infty(\mathbf{D}_{\mathbf{f},\text{rad}}^{\mathbf{m}})$ is a Banach algebra under pointwise multiplication and the norm $\|\cdot\|_\infty$. For each $p\in\mathbb{N}$, we define the norms $\|\cdot\|_p : M_{p\times p}\left(H^\infty(\mathbf{D}_{\mathbf{f},\text{rad}}^{\mathbf{m}})\right)\to[0,\infty)$ by setting

$$\|[\varphi_{st}]_{p\times p}\|_p := \sup \|[\varphi_{st}(X_{i,j})]_{p\times p}\|,$$

where the supremum is taken over all $(X_{i,j})\in \mathbf{D}_{\mathbf{f},\text{rad}}^{\mathbf{m}}(\mathcal{H})$ and any Hilbert space $\mathcal{H}$. It is easy to see that the norms $\|\cdot\|_p$, $p\in\mathbb{N}$, determine an operator space structure on $H^\infty(\mathbf{D}_{\mathbf{f},\text{rad}}^{\mathbf{m}})$, in the sense of Ruan ([31]). Let φ be a free holomorphic function on the abstract radial polydomain $\mathbf{D}_{\mathbf{f},\text{rad}}^{\mathbf{m}}$. Note that if $0<r_1<r_2<1$, then $r_1\mathbf{D}_{\mathbf{f}}^{\mathbf{m}}(\mathcal{H})\subset r_2\mathbf{D}_{\mathbf{f}}^{\mathbf{m}}(\mathcal{H})\subset \mathbf{D}_{\mathbf{f}}^{\mathbf{m}}(\mathcal{H})$. Since $\varphi(r_2\mathbf{W}_{i,j})$ is in the polydomain algebra $\mathcal{A}(\mathbf{D}_{\mathbf{f}}^{\mathbf{m}})$, Theorem 2.4 implies $\|\varphi(rr_2\mathbf{W}_{i,j})\|\leq\|\varphi(r_2\mathbf{W}_{i,j})\|$ for any $r\in[0,1)$. Taking $r:=\frac{r_1}{r_2}$, we deduce that

$$\|\varphi(r_1\mathbf{W}_{i,j})\|\leq\|\varphi(r_2\mathbf{W}_{i,j})\|.$$

On the other hand, if $0<r<1$, then we can use again Theorem 2.4 to show that the mapping $g: r\mathbf{D}_{\mathbf{f}}^{\mathbf{m}}(\mathcal{H})\to B(\mathcal{H})$ defined by

$$g(X_{i,j}) := \varphi(X_{i,j}), \qquad (X_{i,j})\in r\mathbf{D}_{\mathbf{f}}^{\mathbf{m}}(\mathcal{H}),$$

is continuous and $\|g(X_{i,j})\|\leq\|g(r\mathbf{W}_{i,j})\|$. Moreover, the series defining g converges uniformly on $r\mathbf{D}_{\mathbf{f}}^{\mathbf{m}}(\mathcal{H})$ in the operator norm topology.

Given $\varphi \in F^\infty(\mathbf{D}_\mathbf{f}^\mathbf{m})$ and a Hilbert space $\mathcal{H}$, the noncommutative Berezin transform associated with the abstract noncommutative polydomain $\mathbf{D}_\mathbf{f}^\mathbf{m}$ generates a function whose representation on $\mathcal{H}$ is

$$\mathbf{B}[\varphi] : \mathbf{D}_{\mathbf{f},\mathrm{rad}}^\mathbf{m}(\mathcal{H}) \to B(\mathcal{H})$$

defined by

$$\mathbf{B}[\varphi](X_{i,j}) := \mathbf{B}_X[\varphi], \qquad X := (X_{i,j}) \in \mathbf{D}_{\mathbf{f},\mathrm{rad}}^\mathbf{m}(\mathcal{H}),$$

where $\mathbf{B}_X$ is the Berezin transform at X. We call $\mathbf{B}[\varphi]$ the Berezin transform of φ. In what follows, we identify the noncommutative algebra $F^\infty(\mathbf{D}_\mathbf{f}^\mathbf{m})$ with the Hardy subalgebra $H^\infty(\mathbf{D}_{\mathbf{f},\mathrm{rad}}^\mathbf{m})$ of bounded free holomorphic functions on $\mathbf{D}_{\mathbf{f},\mathrm{rad}}^\mathbf{m}$.

Theorem 4.2. *The map $\Phi : H^\infty(\mathbf{D}_{\mathbf{f},\mathrm{rad}}^\mathbf{m}) \to F^\infty(\mathbf{D}_\mathbf{f}^\mathbf{m})$ defined by*

$$\Phi\left(\sum_{(\alpha)} a_{(\alpha)} Z_{(\alpha)}\right) := \sum_{(\alpha)} a_{(\alpha)} \mathbf{W}_{(\alpha)}$$

is a completely isometric isomorphism of operator algebras. Moreover, if $g := \sum_{(\alpha)} a_{(\alpha)} Z_{(\alpha)}$ is a free holomorphic function on the abstract radial polydomain $\mathbf{D}_{\mathbf{f},\mathrm{rad}}^\mathbf{m}$, then the following statements are equivalent:

(i) *$g \in H^\infty(\mathbf{D}_{\mathbf{f},\mathrm{rad}}^\mathbf{m})$;*
(ii) *$\sup_{0\le r<1} \|g(r\mathbf{W}_{i,j})\| < \infty$, where $g(r\mathbf{W}_{i,j}) := \sum_{q=0}^\infty \sum_{\substack{(\alpha)\in \mathbb{F}_{n_1}^+\times\cdots\times\mathbb{F}_{n_k}^+\\ |\alpha_1|+\cdots+|\alpha_k|=q}} r^q a_{(\alpha)} \mathbf{W}_{(\alpha)}$;*
(iii) *there exists $\varphi \in F^\infty(\mathbf{D}_\mathbf{f}^\mathbf{m})$ with $g = \mathbf{B}[\varphi]$, where $\mathbf{B}$ is the noncommutative Berezin transform associated with the abstract polydomain $\mathbf{D}_\mathbf{f}^\mathbf{m}$.*

In this case,

$$\Phi(g) = \text{SOT-}\lim_{r\to 1} g(r\mathbf{W}_{i,j}), \qquad \Phi^{-1}(\varphi) = \mathbf{B}[\varphi], \quad \varphi \in F^\infty(\mathbf{D}_\mathbf{f}^\mathbf{m}),$$

and

$$\|\Phi(g)\| = \sup_{0\le r<1} \|g(r\mathbf{W}_{i,j})\| = \lim_{r\to 1} \|g(r\mathbf{W}_{i,j})\|.$$

Proof. To show that the map Φ is well-defined, let $g := \sum_{(\beta)} a_{(\beta)} Z_{(\beta)}$ be in the Hardy algebra $H^\infty(\mathbf{D}_{\mathbf{f},\mathrm{rad}}^\mathbf{m})$. Since $(r\mathbf{W}_{i,j}) \in \mathbf{D}_{\mathbf{f},\mathrm{rad}}^\mathbf{m}(F^2(H_n))$, Lemma 4.1 shows that $g(r\mathbf{W}_{i,j})$ is well-defined for any $r \in [0,1)$ and $\sup_{0\le r<1} \|g(r\mathbf{W}_{i,j})\| \le \|g\|_\infty < \infty$. We need to show that $g(\mathbf{W}_{i,j}) := \sum_{(\beta)} a_{(\beta)} \mathbf{W}_{(\beta)}$ is the Fourier representation of an element in $F^\infty(\mathbf{D}_\mathbf{f}^\mathbf{m})$. Taking into account relation (2.6), we deduce that

$$\sum_{\beta_1\in\mathbb{F}_{n_1}^+,\ldots,\beta_k\in\mathbb{F}_{n_k}^+} r^{|\beta_1|+\cdots+|\beta_k|} |a_{\beta_1,\ldots,\beta_k}|^2 \frac{1}{b_{1,\beta_1}^{(m_1)}\cdots b_{k,\beta_k}^{(m_k)}} = \|g(r\mathbf{W}_{i,j})(1)\| \le \sup_{0\le r<1} \|g(r\mathbf{W}_{i,j})\| < \infty$$

for any $0 \le r < 1$. Consequently, $\sum_{\beta_1\in\mathbb{F}_{n_1}^+,\ldots,\beta_k\in\mathbb{F}_{n_k}^+} |a_{\beta_1,\ldots,\beta_k}|^2 \frac{1}{b_{1,\beta_1}^{(m_1)}\cdots b_{k,\beta_k}^{(m_k)}} < \infty$. As in Section 3, the latter relation implies that $g(\mathbf{W}_{i,j})p$ is in the tensor product $\otimes_{i=1}^k F^2(H_{n_i})$ for any polynomial $p \in \otimes_{i=1}^k F^2(H_{n_i})$. Now assume that $g(\mathbf{W}_{i,j}) \notin F^\infty(\mathbf{D}_\mathbf{f}^\mathbf{m})$. According to the definition of $F^\infty(\mathbf{D}_\mathbf{f}^\mathbf{m})$, for any fixed positive number M, there exists a polynomial $q \in \otimes_{i=1}^k F^2(H_{n_i})$ with $\|q\| = 1$ such that $\|g(\mathbf{W}_{i,j})q\| > M$. Since $\|g(r\mathbf{W}_{i,j})(1) - g(\mathbf{W}_{i,j})(1)\| \to 0$ as $r \to 1$, we have $\|g(\mathbf{W}_{i,j})q - g(r\mathbf{W}_{i,j})q\| \to 0$, as $r \to 1$. Consequently, there is $r_0 \in (0,1)$ such that $\|g(r_0\mathbf{W}_{i,j})q\| > M$, which implies $\|g(r_0\mathbf{W}_{i,j})\| > M$. This contradicts the fact that $\sup_{0\le r<1} \|g(r\mathbf{W}_{i,j})\| < \infty$. Therefore, $g(\mathbf{W}_{i,j}) \in F^\infty(\mathbf{D}_\mathbf{f}^\mathbf{m})$, which proves that the map Φ is well-defined.

Moreover, due to Theorem 2.4 , we have $\|g(X_{i,j}))\| \le \|g(r\mathbf{W}_{i,j})\|$ for any $(X_{i,j}) \in r\mathbf{D}_\mathbf{f}^\mathbf{m}(\mathcal{H})$. Using now Lemma 3.3, we deduce that

$$\|g(\mathbf{W}_{i,j})\| = \sup_{0\le r<1} \|g(r\mathbf{W}_{i,j}))\| = \|g\|_\infty$$

and

$$\Phi(g) = g(\mathbf{W}_{i,j}) = \text{SOT-}\lim_{r\to 1} g(r\mathbf{W}_{i,j}).$$

Therefore, Φ is a well-defined isometric linear map. We show now that Φ is a surjective map. To this end, let $\varphi(\mathbf{W}_{i,j}):=\sum_{(\beta)} a_{(\beta)}\mathbf{W}_{(\beta)}$ be in $F^\infty(\mathbf{D_f^m})$. Using Lemma 3.3 and Theorem 4.1, we deduce that $g:=\sum_{(\alpha)} a_{(\alpha)}Z_{(\alpha)}$ is a free holomorphic function on the noncommutative domain $\mathbf{D}_{\mathbf{f},\text{rad}}^{\mathbf{m}}$ and

$$\|g(X_{i,j})\|\leq \|g(r\mathbf{W}_{i,j})\|\leq \|g(\mathbf{W}_{i,j})\|$$

for any $(X_{i,j})\in r\mathbf{D_f^m}(\mathcal{H})$ and $r\in[0,1)$. Hence, we deduce that

$$\sup_{(X_{i,j})\in \mathbf{D}_{\mathbf{f},\text{rad}}^{\mathbf{m}}(\mathcal{H})}\|g(X_{i,j})\|\leq \|g(\mathbf{W}_{i,j}))\|<\infty,$$

which proves that $g\in H^\infty(\mathbf{D}_{\mathbf{f},\text{rad}}^{\mathbf{m}})$. This shows that the map Φ is surjective. Therefore, we have proved that Φ is an isometric isomorphism of operator algebras. Using the same techniques and passing to matrices, one can prove that Φ is a completely isometric isomorphism. Moreover, note that if $X:=(X_{i,j})\in \mathbf{D}_{\mathbf{f},\text{rad}}^{\mathbf{m}}$, then there is $r\in(0,1)$ such that $X=rY$ with $Y=(Y_{i,j})\in \mathbf{D_f^m}(\mathcal{H})$. Applying Theorem 3.5 part (iii), we deduce that $\varphi(X)=\mathbf{B}_X[\varphi]$. Now, the equivalences mentioned in the theorem can be easily deduced from the considerations above. The proof is complete. $\square$

For the rest of this section, we assume that $\mathbf{D_f^m}(\mathcal{H})$ is closed in the operator norm topology for any Hilbert space $\mathcal{H}$. Then we have $\mathbf{D}_{\mathbf{f},\text{rad}}^{\mathbf{m}}(\mathcal{H})^-=\mathbf{D_f^m}(\mathcal{H})$. Note that the interior of $\mathbf{D_f^m}(\mathcal{H})$, which we denote by $Int(\mathbf{D_f^m}(\mathcal{H}))$, is a subset of $\mathbf{D}_{\mathbf{f},\text{rad}}^{\mathbf{m}}(\mathcal{H})$. We remark that if $\mathbf{q}=(q_1,\ldots,q_k)$ is a k-tuple of positive regular noncommutative polynomials, then $\mathbf{D_q^m}(\mathcal{H})$ is closed in the operator norm topology.

We denote by $A(\mathbf{D}_{\mathbf{f},\text{rad}}^{\mathbf{m}})$ the set of all elements g in $Hol(\mathbf{D}_{\mathbf{f},\text{rad}}^{\mathbf{m}})$ such that the mapping

$$\mathbf{D}_{\mathbf{f},\text{rad}}^{\mathbf{m}}(\mathcal{H})\ni (X_{i,j})\mapsto g(X_{i,j})\in B(\mathcal{H})$$

has a continuous extension to $[\mathbf{D}_{\mathbf{f},\text{rad}}^{\mathbf{m}}(\mathcal{H})]^-=\mathbf{D_f^m}(\mathcal{H})$ for any Hilbert space $\mathcal{H}$. One can show that $A(\mathbf{D}_{\mathbf{f},\text{rad}}^{\mathbf{m}})$ is a Banach algebra under pointwise multiplication and the norm $\|\cdot\|_\infty$, and it has an operator space structure under the norms $\|\cdot\|_p$, $p\in\mathbb{N}$. Moreover, we can identify the polydomain algebra $\mathcal{A}(\mathbf{D_f^m})$ with the subalgebra $A(\mathbf{D}_{\mathbf{f},\text{rad}}^{\mathbf{m}})$. Using Theorem 2.4, Theorem 4.2, and an approximation argument, one can obtain the following result.

Corollary 4.3. *The map $\Phi: A(\mathbf{D}_{\mathbf{f},\text{rad}}^{\mathbf{m}})\to \mathcal{A}(\mathbf{D_f^m})$ defined by*

$$\Phi\left(\sum_{(\alpha)} a_{(\alpha)}Z_{(\alpha)}\right):=\sum_{(\alpha)} a_{(\alpha)}\mathbf{W}_{(\alpha)}$$

is a completely isometric isomorphism of operator algebras. Moreover, if $g:=\sum_{(\alpha)} a_{(\alpha)}Z_{(\alpha)}$ is a free holomorphic function on the abstract radial polydomain $\mathbf{D}_{\mathbf{f},\text{rad}}^{\mathbf{m}}$, then the following statements are equivalent:

(i) *$g\in A(\mathbf{D}_{\mathbf{f},\text{rad}}^{\mathbf{m}})$;*
(ii) *$g(r\mathbf{W}_{i,j}):=\sum_{q=0}^\infty \sum_{\substack{(\alpha)\in \mathbb{F}_{n_1}^+\times\cdots\times\mathbb{F}_{n_k}^+\\ |\alpha_1|+\cdots+|\alpha_k|=q}} r^q a_{(\alpha)}\mathbf{W}_{(\alpha)}$ is convergent in the norm topology as $r\to 1$;*
(iii) *there exists $\varphi\in\mathcal{A}(\mathbf{D_f^m})$ with $g=\mathbf{B}[\varphi]$, where $\mathbf{B}$ is the noncommutative Berezin transform associated with the abstract polydomain $\mathbf{D}_f^m$.*

In this case,

$$\Phi(g)=\lim_{r\to 1} g(r\mathbf{W}_{i,j})\quad \textit{and}\quad \Phi^{-1}(\varphi)=\mathbf{B}[\varphi],\quad \varphi\in\mathcal{A}(\mathbf{D_f^m}).$$

We remark that there is an important connection between the theory of free holomorphic functions on abstract radial polydomains $\mathbf{D}_{\mathbf{f},\text{rad}}^{\mathbf{m}}$, and the theory of holomorphic functions on polydomains in $\mathbb{C}^d$. Indeed, consider the case when $\mathcal{H}=\mathbb{C}^p$ and $p=1,2\ldots$. Then $\mathbf{D_f^m}(\mathbb{C}^p)$ can be seen as a subset of $\mathbb{C}^{(n_1+\cdots+n_k)p^2}$ with an arbitrary norm. We denote by $Int(\mathbf{D_f^m}(\mathbb{C}^p))$ the interior of the closed set $\mathbf{D_f^m}(\mathbb{C}^p)$. In the particular case when $p=1$, the interior $Int(\mathbf{D_f^m}(\mathbb{C}))$ is a Reinhardt domain, i.e., $(\xi_{i,j}\lambda_{i,j})\in Int(\mathbf{D_f^m}(\mathbb{C}))$ for any $(\lambda_{i,j})\in Int(\mathbf{D_f^m}(\mathbb{C}))$ and $\xi_{i,j}\in\mathbb{T}$. Let $M_{p\times p}(\mathbb{C})$ denote the set of all $p\times p$ matrices with entries in $\mathbb{C}$.

Proposition 4.4. *If $p \in \mathbb{N}$ and φ is a free holomorphic function on the abstract radial polydomain $\mathbf{D}_{\mathbf{f},\mathrm{rad}}^{\mathbf{m}}$, then its representation on $\mathbb{C}^p$, i.e., the map $\widehat{\varphi}$ defined by*

$$\mathbb{C}^{(n_1+\cdots+n_k)p^2} \supset \mathbf{D}_{\mathbf{f},\mathrm{rad}}^{\mathbf{m}}(\mathbb{C}^p) \ni (\lambda_{i,j}) \mapsto \varphi(\lambda_{i,j}) \in M_{p\times p}(\mathbb{C}) \subset \mathbb{C}^{p^2}$$

is a holomorphic function on the interior of $\mathbf{D}_{\mathbf{f}}^{\mathbf{m}}(\mathbb{C}^p)$. Moreover, the following statements hold:

(i) *if $\varphi \in F^\infty(\mathbf{D}_{\mathbf{f},\mathrm{rad}}^{\mathbf{m}})$, then $\widehat{\varphi}$ is bounded on the interior of $\mathbf{D}_{\mathbf{f}}^{\mathbf{m}}(\mathbb{C}^p)$;*

(ii) *if $\varphi \in A(\mathbf{D}_{\mathbf{f},\mathrm{rad}}^{\mathbf{m}})$, then $\widehat{\varphi}$ is continuous on $\mathbf{D}_{\mathbf{f}}^{\mathbf{m}}(\mathbb{C}^p)$ and holomorphic on the interior of $\mathbf{D}_{\mathbf{f}}^{\mathbf{m}}(\mathbb{C}^p)$.*

Proof. If K is a compact subset in the interior of $\mathbf{D}_{\mathbf{f}}^{\mathbf{m}}(\mathbb{C}^p)$, then there exists $r \in (0,1)$ such that $K \subset r\mathbf{D}_{\mathbf{f}}^{\mathbf{m}}(\mathbb{C}^p)$. Indeed, if $\lambda := (\lambda_{i,j}) \in Int(\mathbf{D}_{\mathbf{f}}^{\mathbf{m}}(\mathbb{C}^p)) \subset \mathbb{C}^{(n_1+\cdots+n_k)p^2}$, then there exists $\epsilon_\lambda > 0$ and $r_\lambda \in (0,1)$ such that $\frac{1}{r_\lambda}\mu \in Int(\mathbf{D}_{\mathbf{f}}^{\mathbf{m}}(\mathbb{C}^p))$ for any $\mu \in B_{\epsilon_\lambda}(\lambda) := \{z \in \mathbb{C}^{(n_1+\cdots+n_k)p^2} : \ \|\lambda - z\| < \epsilon_\lambda\}$. Since K is a compact set and $K \subset \cup_{\lambda\in K} B_{\epsilon_\lambda}(\lambda)$, there exists $\lambda_1, \ldots \lambda_l \in K$ such that $K \subset \cup_{i=1}^l B_{\epsilon_{\lambda_i}}(\lambda_i)$. Consequently, for any $\mu \in K$, we have $\frac{1}{r_{\lambda_i}}\mu \in Int(\mathbf{D}_{\mathbf{f}}^{\mathbf{m}}(\mathbb{C}^p)) \subset \mathbf{D}_{\mathbf{f}}^{\mathbf{m}}(\mathbb{C}^p)$ for some $i \in \{1, \ldots, n\}$. Taking into account that $r_1\mathbf{D}_{\mathbf{f}}^{\mathbf{m}}(\mathbb{C}^p) \subset r_2\mathbf{D}_{\mathbf{f}}^{\mathbf{m}}(\mathbb{C}^p)$ if $r_1, r_2 \in (0,1)$ and $r_1 \le r_2$, we conclude that $K \subset r\mathbf{D}_{\mathbf{f}}^{\mathbf{m}}(\mathbb{C}^p)$, where $r := \max\{r_1, \ldots, r_l\}$.

Note that if $\varphi := \sum_{q=0}^\infty \sum_{\substack{(\alpha)\in \mathbb{F}_{n_1}^+\times\cdots\times\mathbb{F}_{n_k}^+ \\ |\alpha_1|+\cdots+|\alpha_k|=q}} a_{(\alpha)} Z_{(\alpha)}$, then

$$\left\|\varphi(\lambda_{i,j}) - \sum_{\substack{(\alpha)\in \mathbb{F}_{n_1}^+\times\cdots\times\mathbb{F}_{n_k}^+ \\ |\alpha_1|+\cdots+|\alpha_k|\le n}} a_{(\alpha)}\lambda_{(\alpha)}\right\| \le \sum_{s=n+1}^\infty \left\|\sum_{\substack{(\alpha)\in \mathbb{F}_{n_1}^+\times\cdots\times\mathbb{F}_{n_k}^+ \\ |\alpha_1|+\cdots+|\alpha_k|=s}} r^{|\alpha_1|+\cdots+|\alpha_k|} a_{(\alpha)}\mathbf{W}_{(\alpha)}\right\|$$

for any $(\lambda_{i,j}) \in K$. Using Theorem 4.1, we deduce that $\sum_{\substack{(\alpha)\in \mathbb{F}_{n_1}^+\times\cdots\times\mathbb{F}_{n_k}^+ \\ |\alpha_1|+\cdots+|\alpha_k|\le n}} a_{(\alpha)}\lambda_{(\alpha)}$ converges to $\varphi(\lambda_{i,j})$ uniformly on K, as $n \to \infty$. Therefore, the map $(\lambda_{i,j}) \mapsto \varphi(\lambda_{i,j})$ is holomorphic on the interior of $\mathbf{D}_{\mathbf{f}}^{\mathbf{m}}(\mathbb{C}^p)$. Now, the items (i) and (ii) are consequences of Theorem 4.2 and Corollary 4.3. The proof is complete. ∎

We remark that one can obtain versions of all the results of this section in the setting of free holomorphic functions with operator-valued coefficients. Since the proofs are very similar we shall omit them. We also mention that, in the particular case when $k = m_1 = 1$ and $f_1 = Z_1 + \cdots + Z_n$, we recover some of the results concerning the free holomorphic functions on the unit ball of $B(\mathcal{H})^n$ (see [40], [45], [47]).

5. Joint invariant subspaces and universal models

We obtain a Beurling type factorization and a characterization of the Beurling [12] type joint invariant subspaces under $\{\mathbf{W}_{i,j}\}$. We also characterize the reducing subspaces under $\{\mathbf{W}_{i,j}\}$ and present several results concerning the model theory for pure elements in the noncommutative polydomain $\mathbf{D}_{\mathbf{f}}^{\mathbf{m}}(\mathcal{H})$.

We recall that a subspace $\mathcal{H} \subseteq \mathcal{K}$ is called co-invariant under $\mathcal{S} \subset B(\mathcal{K})$ if $X^*\mathcal{H} \subseteq \mathcal{H}$ for any $X \in \mathcal{S}$.

Theorem 5.1. *Let $\mathbf{W} := (\mathbf{W}_1, \ldots, \mathbf{W}_k)$ be the universal model associated to the abstract noncommutative domain $\mathbf{D}_{\mathbf{f}}^{\mathbf{m}}$. If $\mathcal{K}$ be a Hilbert space and $\mathcal{M} \subseteq (\otimes_{i=1}^k F^2(H_{n_i})) \otimes \mathcal{K}$ is a co-invariant subspace under each operator $\mathbf{W}_{i,j} \otimes I_{\mathcal{K}}$ for $i \in \{1, \ldots, k\}$, $j \in \{1, \ldots, n_i\}$, then there exists a subspace $\mathcal{E} \subseteq \mathcal{K}$ such that*

$$\overline{\operatorname{span}}\left\{(\mathbf{W}_{1,\beta_1}\cdots\mathbf{W}_{k,\beta_k} \otimes I_{\mathcal{K}})\mathcal{M} : \ \beta_1 \in \mathbb{F}_{n_1}^+, \ldots, \beta_k \in \mathbb{F}_{n_k}^+\right\} = (\otimes_{i=1}^k F^2(H_{n_i})) \otimes \mathcal{E}.$$

Consequently, a subspace $\mathcal{M} \subseteq (\otimes_{i=1}^k F^2(H_{n_i})) \otimes \mathcal{K}$ is reducing under each operator $\mathbf{W}_{i,j} \otimes I_{\mathcal{K}}$ for $i \in \{1, \ldots, k\}$, $j \in \{1, \ldots, n_i\}$, if and only if there exists a subspace $\mathcal{E} \subseteq \mathcal{K}$ such that

$$\mathcal{M} = (\otimes_{i=1}^k F^2(H_{n_i})) \otimes \mathcal{E}.$$

Proof. Define the subspace $\mathcal{E} \subseteq \mathcal{K}$ by $\mathcal{E} := (\mathbf{P}_{\mathbb{C}} \otimes I_{\mathcal{K}})\mathcal{M}$, where $\mathbf{P}_{\mathbb{C}}$ is the orthogonal projection from $\otimes_{i=1}^k F^2(H_{n_i})$ onto $\mathbb{C}1 \subset \otimes_{i=1}^k F^2(H_{n_i})$. Let φ be a nonzero element of $\mathcal{M}$ with representation

$$\varphi = \sum_{\beta_1\in\mathbb{F}_{n_1}^+,\ldots,\beta_k\in\mathbb{F}_{n_k}^+} e_{\beta_1}^1 \otimes \cdots \otimes e_{\beta_k}^k \otimes h_{\beta_1,\ldots,\beta_k},$$

where $h_{\beta_1,\ldots,\beta_k} \in \mathcal{K}$ and $\sum_{\beta_1\in\mathbb{F}_{n_1}^+,\ldots,\beta_k\in\mathbb{F}_{n_k}^+} \|h_{\beta_1,\ldots,\beta_k}\|^2 < \infty$. Let $\sigma_1 \in \mathbb{F}_{n_1}^+,\ldots,\sigma_k \in \mathbb{F}_{n_k}^+$ be such that $h_{\sigma_1,\ldots,\sigma_k} \neq 0$ and note that

$$(\mathbf{P}_{\mathbb{C}} \otimes I_{\mathcal{K}})(\mathbf{W}_{1,\sigma_1}^* \cdots \mathbf{W}_{k,\sigma_k}^* \otimes I_{\mathcal{K}})\varphi = 1 \otimes \frac{1}{\sqrt{b_{1,\sigma_1}^{(m_1)}}} \cdots \frac{1}{\sqrt{b_{k,\sigma_k}^{(m_k)}}} h_{\sigma_1,\ldots,\sigma_k}.$$

Consequently, since $\mathcal{M}$ is a co-invariant subspace under $\mathbf{W}_{i,j} \otimes I_{\mathcal{K}}$ for $i \in \{1,\ldots,k\}$, $j \in \{1,\ldots,n_i\}$, we deduce that $h_{\sigma_1,\ldots,\sigma_k} \in \mathcal{E}$. This implies

$$(\mathbf{W}_{1,\sigma_1} \cdots \mathbf{W}_{k,\sigma_k} \otimes I_{\mathcal{K}})(1 \otimes h_{\sigma_1,\ldots,\sigma_k}) = \frac{1}{\sqrt{b_{1,\sigma_1}^{(m_1)}}} \cdots \frac{1}{\sqrt{b_{k,\sigma_k}^{(m_k)}}} e_{\sigma_1}^1 \otimes \cdots \otimes e_{\sigma_k}^k \otimes h_{\sigma_1,\ldots,\sigma_k}$$

is a vector in $\otimes_{i=1}^k F^2(H_{n_i}) \otimes \mathcal{E}$. Therefore,

$$\varphi = \lim_{n\to\infty} \sum_{q=0}^{n} \sum_{\substack{\beta_1\in\mathbb{F}_{n_1}^+,\ldots,\beta_k\in\mathbb{F}_{n_k}^+ \\ |\beta_1|+\cdots+|\beta_k|=q}} e_{\beta_1}^1 \otimes \cdots \otimes e_{\beta_k}^k \otimes h_{\beta_1,\ldots,\beta_k} \tag{5.1}$$

is in $\otimes_{i=1}^k F^2(H_{n_i}) \otimes \mathcal{E}$. Hence, $\mathcal{M} \subset \otimes_{i=1}^k F^2(H_{n_i}) \otimes \mathcal{E}$ and

$$\mathcal{Y} := \overline{\text{span}}\left\{(\mathbf{W}_{1,\sigma_1} \cdots \mathbf{W}_{k,\sigma_k} \otimes I_{\mathcal{K}})\mathcal{M} :\ \sigma_1 \in \mathbb{F}_{n_1}^+,\ldots,\sigma_k \in \mathbb{F}_{n_k}^+\right\} \subset (\otimes_{i=1}^k F^2(H_{n_i})) \otimes \mathcal{E}.$$

To prove the reverse inclusion, we show first that $\mathcal{E} \subset \mathcal{Y}$. If $h_0 \in \mathcal{E}$, $h_0 \neq 0$, then there exists $g \in \mathcal{M} \subset \otimes_{i=1}^k F^2(H_{n_i}) \otimes \mathcal{E}$ such that

$$g = 1 \otimes h_0 + \sum_{\substack{\beta_1\in\mathbb{F}_{n_1}^+,\ldots,\beta_k\in\mathbb{F}_{n_k}^+ \\ |\beta_1|+\cdots+|\beta_k|\geq 1}} e_{\beta_1}^1 \otimes \cdots \otimes e_{\beta_k}^k \otimes h_{\beta_1,\ldots,\beta_k}$$

and $1 \otimes h_0 = (\mathbf{P}_{\mathbb{C}} \otimes I_{\mathcal{K}})g$. Consequently, due to Lemma 2.1, we have

$$1 \otimes h_0 = (\mathbf{P}_{\mathbb{C}} \otimes I_{\mathcal{K}})g = (I - \Phi_{q_1,\mathbf{W}_1\otimes I_{\mathcal{K}}})^{m_1} \cdots (I - \Phi_{q_k,\mathbf{W}_k\otimes I_{\mathcal{K}}})^{m_k}(I)g.$$

Taking into account that $\mathcal{M}$ is co-invariant under $\mathbf{W}_{i,j} \otimes I_{\mathcal{K}}$ for $i \in \{1,\ldots,k\}$, $j \in \{1,\ldots,n_i\}$, we deduce that $h_0 \in \mathcal{Y}$ for any $h_0 \in \mathcal{E}$, i.e., $\mathcal{E} \subset \mathcal{Y}$. The latter inclusion shows that $(\mathbf{W}_{1,\sigma_1} \cdots \mathbf{W}_{k,\sigma_k} \otimes I_{\mathcal{K}})(1\otimes\mathcal{E}) \subset \mathcal{Y}$ for any $\sigma_1 \in \mathbb{F}_{n_1}^+,\ldots,\sigma_k \in \mathbb{F}_{n_k}^+$, which implies

$$\frac{1}{\sqrt{b_{1,\sigma_1}^{(m_1)}}} \cdots \frac{1}{\sqrt{b_{k,\sigma_k}^{(m_k)}}} e_{\sigma_1}^1 \otimes \cdots \otimes e_{\sigma_k}^k \otimes \mathcal{E} \subset \mathcal{Y}.$$

Hence, if $\varphi \in (\otimes_{i=1}^k F^2(H_{n_i})) \otimes \mathcal{E}$ has the representation (5.1), we deduce that $\varphi \in \mathcal{Y}$. Therefore, $(\otimes_{i=1}^k F^2(H_{n_i})) \otimes \mathcal{E} \subseteq \mathcal{Y}$. The last part of the theorem is now obvious. The proof is complete. □

Let $\mathbf{W} := (\mathbf{W}_1,\ldots,\mathbf{W}_k)$ be the universal model associated to the abstract noncommutative domain $\mathbf{D}_{\mathbf{f}}^{\mathbf{m}}$. An operator $M : (\otimes_{i=1}^k F^2(H_{n_i})) \otimes \mathcal{H} \to (\otimes_{i=1}^k F^2(H_{n_i})) \otimes \mathcal{K}$ is called *multi-analytic* with respect to $\mathbf{W}$ if

$$M(\mathbf{W}_{i,j} \otimes I_{\mathcal{H}}) = (\mathbf{W}_{i,j} \otimes I_{\mathcal{K}})M$$

for any $i \in \{1,\ldots,k\}$ and $j \in \{1,\ldots,n_i\}$. In case M is a partial isometry, we call it *inner multi-analytic* operator.

Theorem 5.2. *Let $\mathbf{W} := (\mathbf{W}_1,\ldots,\mathbf{W}_k)$ be the universal model associated to the abstract noncommutative domain $\mathbf{D}_{\mathbf{f}}^{\mathbf{m}}$ and let $\mathbf{W}_i \otimes I_{\mathcal{H}} := (\mathbf{W}_{i,1} \otimes I_{\mathcal{H}},\ldots,\mathbf{W}_{i,n_i} \otimes I_{\mathcal{H}})$ for $i \in \{1,\ldots,k\}$, where $\mathcal{H}$ is a Hilbert space. If $Y \in B((\otimes_{i=1}^k F^2(H_{n_i})) \otimes \mathcal{H})$ then the following statements are equivalent.*

(i) *There is a multi-analytic operator $M : (\otimes_{i=1}^k F^2(H_{n_i})) \otimes \mathcal{E} \to (\otimes_{i=1}^k F^2(H_{n_i})) \otimes \mathcal{H}$ with respect to $\mathbf{W}$, where $\mathcal{E}$ is a Hilbert space, such that*

$$Y = MM^*.$$

(ii) *For any $\mathbf{p} := (p_1,\ldots,p_k) \in \mathbb{Z}_+^k$ such that $\mathbf{p} \leq \mathbf{m}$, $\mathbf{p} \neq 0$,*

$$(id - \Phi_{f_1,\mathbf{W}_1\otimes I_{\mathcal{H}}})^{p_1} \cdots (id - \Phi_{f_k,\mathbf{W}_k\otimes I_{\mathcal{H}}})^{p_k}(Y) \geq 0.$$

Proof. Setting $\mathbf{\Delta}^{\mathbf{p}}_{\mathbf{f},\mathbf{W}\otimes I_\mathcal{H}} := (id - \Phi_{f_1,\mathbf{W}_1\otimes I_\mathcal{H}})^{p_1}\cdots(id - \Phi_{f_k,\mathbf{W}_k\otimes I_\mathcal{H}})^{p_k}$, it is easy to see that if item (i) holds, then

$$\mathbf{\Delta}^{\mathbf{p}}_{\mathbf{f},\mathbf{W}\otimes I_\mathcal{H}}(Y) = M\mathbf{\Delta}^{\mathbf{p}}_{\mathbf{f},\mathbf{W}\otimes I_\mathcal{E}}(I)M^* \geq 0$$

for any $\mathbf{p} := (p_1,\ldots,p_k)\in\mathbb{Z}_+^k$ such that $\mathbf{p}\leq\mathbf{m}$, $\mathbf{p}\neq 0$.

To prove the converse, assume that (ii) holds. In particular, we have $\Phi_{f_1,\mathbf{W}_1\otimes I_\mathcal{H}}(\mathbf{\Delta}^{\mathbf{m}'}_{\mathbf{f},\mathbf{W}\otimes I_\mathcal{H}}(Y)) \leq \mathbf{\Delta}^{\mathbf{m}'}_{\mathbf{f},\mathbf{W}\otimes I_\mathcal{H}}(Y)$, where $\mathbf{m}' = (m_1-1, m_2,\ldots,m_k)$. Consequently, $\Phi^n_{f_1,\mathbf{W}_1\otimes I_\mathcal{H}}(\mathbf{\Delta}^{\mathbf{m}'}_{\mathbf{f},\mathbf{W}\otimes I_\mathcal{H}}(Y)) \leq \mathbf{\Delta}^{\mathbf{m}'}_{\mathbf{f},\mathbf{W}\otimes I_\mathcal{H}}(Y)$ for any $n\in\mathbb{N}$. Since $\mathbf{W} := (\mathbf{W}_1,\ldots,\mathbf{W}_k)$ is a pure k-tuple, we have $\text{SOT-}\lim_{n\to\infty}\Phi^n_{f_1,\mathbf{W}_1\otimes I_\mathcal{H}}(\mathbf{\Delta}^{\mathbf{m}'}_{\mathbf{f},\mathbf{W}\otimes I_\mathcal{H}}(Y)) = 0$, which implies $\mathbf{\Delta}^{\mathbf{m}'}_{\mathbf{f},\mathbf{W}\otimes I_\mathcal{H}}(Y)\geq 0$. Continuing this process, we deduce that $Y\geq 0$.

Let $\mathcal{M} := \overline{\operatorname{range} Y^{1/2}}$ and define

$$A_{i,j}(Y^{1/2}x) := Y^{1/2}(\mathbf{W}^*_{i,j}\otimes I_\mathcal{H})x, \quad x\in(\otimes_{i=1}^k F^2(H_{n_i}))\otimes\mathcal{H}, \tag{5.2}$$

for any $i\in\{1,\ldots,k\}$ and $j\in\{1,\ldots,n_i\}$. Since $\Phi_{f_i,\mathbf{W}_i\otimes I_\mathcal{H}}(Y)\leq Y$, we have

$$\sum_{\alpha\in\mathbb{F}^+_{n_i},|\alpha|\geq 1} a_{i,\alpha}\|A_{i,\tilde\alpha}Y^{1/2}x\|^2 = \langle\Phi_{f_i,\mathbf{W}_i\otimes I_\mathcal{H}}(Y)x,x\rangle \leq \|Y^{1/2}x\|^2$$

for any $x\in(\otimes_{i=1}^k F^2(H_{n_i}))\otimes\mathcal{H}$. Consequently, we deduce that $a_{i,g^i_j}\|A_{i,j}Y^{1/2}x\|^2\leq\|Y^{1/2}x\|^2$, for any $x\in(\otimes_{i=1}^k F^2(H_{n_i}))\otimes\mathcal{H}$. Since $a_{i,g^i_j}\neq 0$ each $A_{i,j}$ can be uniquely be extended to a bounded operator (also denoted by $A_{i,j}$) on the subspace $\mathcal{M}$. Denoting $X_{i,j} := A^*_{i,j}$ for $i\in\{1,\ldots,k\}$, $j\in\{1,\ldots,n_i\}$, an approximation argument shows that $\Phi_{f_i,X_i}(I_\mathcal{M})\leq I_\mathcal{M}$ and relation (5.2) implies

$$X^*_{i,j}(Y^{1/2}x) = Y^{1/2}(\mathbf{W}^*_{i,j}\otimes I_\mathcal{H})x, \qquad x\in(\otimes_{i=1}^k F^2(H_{n_i}))\otimes\mathcal{H},$$

for any $i\in\{1,\ldots,k\}$ and $j\in\{1,\ldots,n_i\}$. This implies

$$Y^{1/2}\mathbf{\Delta}^{\mathbf{p}}_{\mathbf{f},\mathbf{X}}(I_\mathcal{M})Y^{1/2} = \mathbf{\Delta}^{\mathbf{p}}_{\mathbf{f},\mathbf{W}\otimes I_\mathcal{H}}(Y)\geq 0$$

for any $\mathbf{p} := (p_1,\ldots,p_k)\in\mathbb{Z}_+^k$ such that $\mathbf{p}\leq\mathbf{m}$, $\mathbf{p}\neq 0$. On the other hand, we have

$$\left\langle\Phi^n_{f_i,X_i}(I_\mathcal{M})Y^{1/2}x, Y^{1/2}x\right\rangle = \left\langle\Phi^n_{f_i,\mathbf{W}_i\otimes I_\mathcal{H}}(Y)x,x\right\rangle \leq \|Y\|\left\langle\Phi^n_{f_i,\mathbf{W}_i\otimes I_\mathcal{H}}(I)x,x\right\rangle$$

for any $x\in(\otimes_{i=1}^k F^2(H_{n_i}))\otimes\mathcal{H}$ and $n\in\mathbb{N}$. Since $\text{SOT-}\lim\limits_{n\to\infty}\Phi^n_{f_i,\mathbf{W}_i\otimes I_\mathcal{H}}(I) = 0$, we have $\text{SOT-}\lim\limits_{m\to\infty}\Phi^n_{f_i,X_i}(I_\mathcal{M}) = 0$, which, due to Proposition 1.8 shows that $\mathbf{X} := (X_1,\ldots,X_k)$ is a pure k-tuple in the noncommutative polydomain $\mathbf{D}^{\mathbf{m}}_{\mathbf{f}}(\mathcal{M})$. Set $\mathcal{E} := \overline{\mathbf{\Delta}^{\mathbf{m}}_{\mathbf{f},\mathbf{X}}(I_\mathcal{M})(\mathcal{M})}$. According to Theorem 2.2, the noncommutative Berezin kernel $\mathbf{K}_{\mathbf{f},\mathbf{X}} : \mathcal{M}\to(\otimes_{i=1}^k F^2(H_{n_i}))\otimes\mathcal{E}$ is an isometry with the property that

$$X_{i,j}\mathbf{K}^*_{\mathbf{f},\mathbf{X}} = \mathbf{K}^*_{\mathbf{f},\mathbf{X}}(\mathbf{W}_{i,j}\otimes I_\mathcal{E})$$

for any $i\in\{1,\ldots,k\}$ and any $j\in\{1,\ldots,n_i\}$. Now, define the bounded linear operator $M := Y^{1/2}\mathbf{K}^*_{\mathbf{f},\mathbf{X}} : (\otimes_{i=1}^k F^2(H_{n_i}))\otimes\mathcal{E}\to(\otimes_{i=1}^k F^2(H_{n_i}))\otimes\mathcal{H}$ and note that

$$\begin{aligned} M(\mathbf{W}_{i,j}\otimes I_\mathcal{E}) &= Y^{1/2}\mathbf{K}^*_{\mathbf{f},\mathbf{X}}(\mathbf{W}_{i,j}\otimes I_\mathcal{E}) = Y^{1/2}X_{i,j}\mathbf{K}^*_{\mathbf{f},\mathbf{X}} \\ &= (\mathbf{W}_{i,j}\otimes I_\mathcal{H})Y^{1/2}\mathbf{K}^*_{\mathbf{f},\mathbf{X}} = (\mathbf{W}_{i,j}\otimes I_\mathcal{H})M \end{aligned}$$

for any $i\in\{1,\ldots,k\}$ and $j\in\{1,\ldots,n_i\}$, which proves that M is a multi-analytic operator with respect to $\mathbf{W}_{i,j}$. We also have $MM^* = Y^{1/2}\mathbf{K}^*_{\mathbf{f},\mathbf{X}}\mathbf{K}_{\mathbf{f},\mathbf{X}}Y^{1/2} = Y$. This completes the proof. ☐

We say that $\mathcal{M}\subset(\otimes_{i=1}^k F^2(H_{n_i}))\otimes\mathcal{H}$ is a Beurling type invariant subspace under the operators $\mathbf{W}_{i,j}\otimes I_\mathcal{H}$, $i\in\{1,\ldots,k\}$, $j\in\{1,\ldots,n_i\}$, if there is an inner multi-analytic operator with respect to $\mathbf{W}$,

$$\Psi : (\otimes_{i=1}^k F^2(H_{n_i}))\otimes\mathcal{E}\to(\otimes_{i=1}^k F^2(H_{n_i}))\otimes\mathcal{H},$$

such that $\mathcal{M} = \Psi\left((\otimes_{i=1}^k F^2(H_{n_i}))\otimes\mathcal{E}\right)$.

Corollary 5.3. *Let $\mathcal{M}\subset(\otimes_{i=1}^k F^2(H_{n_i}))\otimes\mathcal{H}$ be an invariant subspace under the operators $\mathbf{W}_{i,j}\otimes I_\mathcal{H}$, $i\in\{1,\ldots,k\}$, $j\in\{1,\ldots,n_i\}$. Then $\mathcal{M}$ is of Beurling type if and only if*

$$(id-\Phi_{f_1,\mathbf{W}_1\otimes I_\mathcal{H}})^{p_1}\cdots(id-\Phi_{f_k,\mathbf{W}_k\otimes I_\mathcal{H}})^{p_k}(P_\mathcal{M})\geq 0$$

for any $\mathbf{p}:=(p_1,\ldots,p_k)\in\mathbb{Z}_+^k$ such that $\mathbf{p}\leq\mathbf{m}$, where $P_\mathcal{M}$ is the orthogonal projection of the Hilbert space $(\otimes_{i=1}^k F^2(H_{n_i}))\otimes\mathcal{H}$ onto $\mathcal{M}$. In the particular case when $\mathbf{m}=(1,\ldots,1)$, the condition above is satisfied when $\mathbf{W}\otimes I_\mathcal{H}|_\mathcal{M}:=(\mathbf{W}_1\otimes I_\mathcal{H}|_\mathcal{M},\ldots,\mathbf{W}_k\otimes I_\mathcal{H}|_\mathcal{M})$ is doubly commuting.

Proof. If $\Psi:(\otimes_{i=1}^k F^2(H_{n_i}))\otimes\mathcal{E}\to(\otimes_{i=1}^k F^2(H_{n_i}))\otimes\mathcal{H}$ is a inner multi-analytic operator and $\mathcal{M}=\Psi\left((\otimes_{i=1}^k F^2(H_{n_i}))\otimes\mathcal{E}\right)$, then $P_\mathcal{M}=\Psi\Psi^*$. Taking into account Lemma 2.1, we deduce that

$$(id-\Phi_{f_1,\mathbf{W}_1\otimes I_\mathcal{H}})^{p_1}\cdots(id-\Phi_{f_k,\mathbf{W}_k\otimes I_\mathcal{H}})^{p_k}(P_\mathcal{M})=\Psi(P_\mathbf{C}\otimes I_\mathcal{E})\Psi^*\geq 0$$

for any $\mathbf{p}:=(p_1,\ldots,p_k)\in\mathbb{Z}_+^k$ such that $\mathbf{p}\leq\mathbf{m}$. The converse is a consequence of Theorem 5.2, when we take $Y=P_\mathcal{M}$.

Now, we consider the case when $\mathbf{m}=(1,\ldots,1)$. Note that if $\mathcal{M}$ is an invariant subspace under the operators $\mathbf{W}_{i,j}\otimes I_\mathcal{H}$, then $\mathbf{W}\otimes I_\mathcal{H}|_\mathcal{M}$ is doubly commuting if and only if $P_\mathcal{M}(\mathbf{W}_{i_1,j_1}\otimes I_\mathcal{H})P_\mathcal{M}$ commutes with $P_\mathcal{M}(\mathbf{W}_{i_2,j_2}^*\otimes I_\mathcal{H})P_\mathcal{M}$ for any $i_1,i_2\in\{1,\ldots,k\}$, $i_1\neq i_2$, and any $j_1\in\{1,\ldots,n_{i_1}\}$, $j_2\in\{1,\ldots,n_{i_2}\}$. The latter condition is equivalent to

$$P_\mathcal{M}(\mathbf{W}_{i_1,\alpha}\otimes I_\mathcal{H})P_\mathcal{M}\quad\text{commutes with}\quad P_\mathcal{M}(\mathbf{W}_{i_2,\beta}^*\otimes I_\mathcal{H})P_\mathcal{M}\tag{5.3}$$

for any $\alpha\in\mathbb{F}_{n_{i_1}}^+$ and $\beta\in\mathbb{F}_{n_{i_2}}^+$. Assume that $\mathcal{M}$ is invariant subspace under the operators $\mathbf{W}_{i,j}\otimes I_\mathcal{H}$ and $\mathbf{W}\otimes I_\mathcal{H}|_\mathcal{M}$ is doubly commuting. Then, due to relation (5.3), for any $\alpha_i\in\mathbb{F}_{n_i}^+$, $i\in\{1,\ldots,k\}$, we have

$$\begin{aligned}&(\mathbf{W}_{1,\alpha_1}\otimes I_\mathcal{H})\cdots(\mathbf{W}_{k,\alpha_k}\otimes I_\mathcal{H})P_\mathcal{M}(\mathbf{W}_{k,\alpha_k}^*\otimes I_\mathcal{H})\cdots(\mathbf{W}_{1,\alpha_1}^*\otimes I_\mathcal{H})\\&\qquad=(\mathbf{W}_{1,\alpha_1}\otimes I_\mathcal{H})P_\mathcal{M}(\mathbf{W}_{1,\alpha_1}^*\otimes I_\mathcal{H})\cdots(\mathbf{W}_{k,\alpha_k}\otimes I_\mathcal{H})P_\mathcal{M}(\mathbf{W}_{k,\alpha_k}^*\otimes I_\mathcal{H}).\end{aligned}\tag{5.4}$$

Consequently, we deduce that

$$(id-\Phi_{f_1,\mathbf{W}_1\otimes I_\mathcal{H}})^{p_1}\cdots(id-\Phi_{f_k,\mathbf{W}_k\otimes I_\mathcal{H}})^{p_k}(P_\mathcal{M})=(P_\mathcal{M}-\Phi_{f_1,\mathbf{W}_1\otimes I_\mathcal{H}}(P_\mathcal{M}))^{p_1}\cdots(P_\mathcal{M}-\Phi_{f_k,\mathbf{W}_k\otimes I_\mathcal{H}}(P_\mathcal{M}))^{p_k}$$

for any $\mathbf{p}:=(p_1,\ldots,p_k)\in\mathbb{Z}_+^k$ such that $\mathbf{p}\leq(1,\ldots,1)$. Now, since $\mathbf{W}_1,\ldots,\mathbf{W}_k$ are commuting tuples, we deduce that $P_\mathcal{M}-\Phi_{f_i,\mathbf{W}_i\otimes I_\mathcal{H}}(P_\mathcal{M})$, $i\in\{1,\ldots,k\}$, are commuting operators. On the other hand, they are also positive operators. Indeed, let $\{a_{i,\alpha_i}\}_{\alpha\in\mathbb{F}_{n_i}^+}$ be the coefficients of the positive regular free holomorphic function f_i, and let $x\in(\otimes_{i=1}^k F^2(H_{n_i}))\otimes\mathcal{H}$ have the representation $x=x_1+x_2$ with respect to the orthogonal decomposition $\mathcal{M}\oplus\mathcal{M}^\perp$. Note that

$$\begin{aligned}\langle\Phi_{f_i,\mathbf{W}_i\otimes I_\mathcal{H}}(P_\mathcal{M})x,x\rangle&=\langle\Phi_{f_i,\mathbf{W}_i\otimes I_\mathcal{H}}(P_\mathcal{M})x_1,x_1\rangle=\sum_{|\alpha|\geq 1}a_{i,\alpha_i}\|P_\mathcal{M}(\mathbf{W}_{i,\alpha_i}\otimes I_\mathcal{H})x_1\|^2\\&\leq\langle\Phi_{f_i,\mathbf{W}_i\otimes I_\mathcal{H}}(I)x_1,x_1\rangle\leq\|x_1\|^2=\langle P_\mathcal{M}x,x\rangle,\end{aligned}$$

which proves our assertion. Therefore, we can deduce that

$$(id-\Phi_{f_1,\mathbf{W}_1\otimes I_\mathcal{H}})^{p_1}\cdots(id-\Phi_{f_k,\mathbf{W}_k\otimes I_\mathcal{H}})^{p_k}(P_\mathcal{M})\geq 0$$

for any $\mathbf{p}:=(p_1,\ldots,p_k)\in\mathbb{Z}_+^k$ such that $\mathbf{p}\leq(1,\ldots,1)$. Due to the first part of this corollary, we conclude that $\mathcal{M}$ is a Beurling type invariant subspace under the operators $\mathbf{W}_{i,j}\otimes I_\mathcal{H}$. The proof is complete. ∎

Let $\mathbf{W}:=(\mathbf{W}_1,\ldots,\mathbf{W}_k)$ be the universal model associated to the abstract noncommutative domain $\mathbf{D}_\mathbf{f}^\mathbf{m}$, and let $\Phi:(\otimes_{i=1}^k F^2(H_{n_i}))\otimes\mathcal{H}\to(\otimes_{i=1}^k F^2(H_{n_i}))\otimes\mathcal{K}$ be a multi-analytic operator with respect to $\mathbf{W}$, i.e., if $\Phi(\mathbf{W}_{i,j}\otimes I_\mathcal{H})=(\mathbf{W}_{i,j}\otimes I_\mathcal{K})\Phi$ for any $i\in\{1,\ldots,k\}$ and $j\in\{1,\ldots,n_i\}$. We introduce the *support* of Φ as the smallest reducing subspace $\operatorname{supp}(\Phi)\subset\otimes_{i=1}^k F^2(H_{n_i}))\otimes\mathcal{H}$ under each operator $\mathbf{W}_{i,j}$, containing the co-invariant subspace $\mathcal{M}:=\overline{\Phi^*((\otimes_{i=1}^k F^2(H_{n_i}))\otimes\mathcal{K})}$. Using Theorem 5.1 and its proof, we deduce that

$$\operatorname{supp}(\Phi)=\bigvee_{(\alpha)\in\mathbb{F}_{n_1}^+\times\cdots\times\mathbb{F}_{n_k}^+}(\mathbf{W}_{(\alpha)}\otimes I_\mathcal{H})(\mathcal{M})=(\otimes_{i=1}^k F^2(H_{n_i}))\otimes\mathcal{L},$$

where $\mathcal{L}:=(\mathbf{P}_\mathbb{C}\otimes I_\mathcal{H})\overline{\Phi^*((\otimes_{i=1}^k F^2(H_{n_i}))\otimes\mathcal{K})}$.

Assume that $\mathbf{W} := (\mathbf{W}_1, \dots, \mathbf{W}_k)$ is the universal model associated to the abstract noncommutative domain $\mathbf{D}_{\mathbf{f}}^{\mathbf{m}}$. We remark that if $\Psi : (\otimes_{i=1}^k F^2(H_{n_i})) \otimes \mathcal{E} \to (\otimes_{i=1}^k F^2(H_{n_i})) \otimes \mathcal{H}$ is an isometric multi-analytic operator and $\mathcal{M} = \Psi\left((\otimes_{i=1}^k F^2(H_{n_i})) \otimes \mathcal{E}\right)$, then $\mathbf{W} \otimes I_{\mathcal{H}}|_{\mathcal{M}}$ is doubly commuting. Since this is a straightforward computation, we omit it. The converse of this implication holds true for the noncommutative polyball.

Corollary 5.4. *Let $\mathbf{W} := (\mathbf{W}_1, \dots, \mathbf{W}_k)$ be the universal model associated to the noncommutative polyball $[B(\mathcal{H})^{n_1}]_1^- \times_c \cdots \times_c [B(\mathcal{H})^{n_k}]_1^-$, i.e., $\mathbf{m} = (1, \dots, 1)$ and $f_i := Z_{i,1} + \cdots + Z_{i,n_i}$ for $i \in \{1, \dots, k\}$. If $\mathcal{M} \subset (\otimes_{i=1}^k F^2(H_{n_i})) \otimes \mathcal{H}$ is a nonzero invariant subspace under the operators $\mathbf{W}_{i,j} \otimes I_{\mathcal{H}}$, then $\mathbf{W} \otimes I_{\mathcal{H}}|_{\mathcal{M}}$ is doubly commuting if and only if there is a Hilbert space $\mathcal{L}$ and an isometric multi-analytic operator $\Phi : (\otimes_{i=1}^k F^2(H_{n_i})) \otimes \mathcal{L} \to (\otimes_{i=1}^k F^2(H_{n_i})) \otimes \mathcal{H}$ such that $\mathcal{M} = \Phi((\otimes_{i=1}^k F^2(H_{n_i})) \otimes \mathcal{L})$.*

Proof. Due to the remarks preceding this corollary, it remains to prove the direct implication. Assume that $\mathbf{W} \otimes I_{\mathcal{H}}|_{\mathcal{M}}$ is doubly commuting. Corollary 5.3 and Theorem 5.2 imply the existence of an inner multi-analytic operator $\Psi : (\otimes_{i=1}^k F^2(H_{n_i})) \otimes \mathcal{E} \to (\otimes_{i=1}^k F^2(H_{n_i})) \otimes \mathcal{H}$ such that $\mathcal{M} = \Psi((\otimes_{i=1}^k F^2(H_{n_i})) \otimes \mathcal{E})$. Since $\mathbf{W}_{i,j}$ are isometries, the initial space of Ψ, i.e., $\Psi^*((\otimes_{i=1}^k F^2(H_{n_i})) \otimes \mathcal{H}) = \{x \in (\otimes_{i=1}^k F^2(H_{n_i})) \otimes \mathcal{E} : \|\Psi x\| = \|x\|\}$ is reducing under each $\mathbf{W}_{i,j}$. On the other hand, the *support* of Ψ is the the smallest reducing subspace $\operatorname{supp}(\Psi) \subset F^2(H_{n_i})) \otimes \mathcal{H}$ under each operator $\mathbf{W}_{i,j}$, containing the co-invariant subspace $\Psi^*((\otimes_{i=1}^k F^2(H_{n_i})) \otimes \mathcal{H})$. Therefore, we must have $\operatorname{supp}(\Psi) = \Psi^*((\otimes_{i=1}^k F^2(H_{n_i})) \otimes \mathcal{H})$. Note that $\Phi := \Psi|_{\operatorname{supp}(\Psi)}$ is an isometric multi-analytic operator. Since $\operatorname{supp}(\Psi) = (\otimes_{i=1}^k F^2(H_{n_i})) \otimes \mathcal{L}$, where $\mathcal{L} := (\mathbf{P}_{\mathbb{C}} \otimes I_{\mathcal{E}})\Psi^*((\otimes_{i=1}^k F^2(H_{n_i})) \otimes \mathcal{H})$ and $\mathcal{M} = \Phi((\otimes_{i=1}^k F^2(H_{n_i})) \otimes \mathcal{L})$, the proof is complete. $\square$

We remark that in the particular case when $n_1 = \cdots = n_k = 1$, Corollary 5.4 is a Beurling type result for the the Hardy space $H^2(\mathbb{D}^k)$ of the polydisc, which seems to be new if $k > 2$.

We recall that $\mathcal{P}(\mathbf{W})$ is the set of all polynomials $p(\mathbf{W}_{i,j})$ in the operators $\mathbf{W}_{i,j}$, $i \in \{1, \dots, k\}$, $j \in \{1, \dots, n_i\}$, and the identity.

Lemma 5.5. *If $\mathbf{W} := (\mathbf{W}_1, \dots, \mathbf{W}_k)$ is the universal model associated to the abstract noncommutative polydomain $\mathbf{D}_{\mathbf{f}}^{\mathbf{m}}$, then the C^*-algebra $C^*(\mathbf{W}_{i,j})$ is irreducible.*

Proof. Let $\mathcal{M} \neq \{0\}$ be a subspace of $\otimes_{i=1}^k F^2(H_{n_i})$, which is jointly reducing for each operator $\mathbf{W}_{i,j}$ for all $i \in \{1, \dots, k\}$ and $j \in \{1, \dots, n_i\}$. Let $\varphi \in \mathcal{M}$, $\varphi \neq 0$, and assume that

$$\varphi = \sum_{\beta_1 \in \mathbb{F}_{n_1}^+, \dots, \beta_k \in \mathbb{F}_{n_k}^+} a_{\beta_1, \dots, \beta_k} e_{\beta_1}^1 \otimes \cdots \otimes e_{\beta_k}^k.$$

If $a_{\beta_1, \dots, \beta_k}$ is a nonzero coefficient of φ, then, using relation (2.6), we deduce that

$$\mathbf{P}_{\mathbb{C}} \mathbf{W}_{1,\beta_1}^* \cdots \mathbf{W}_{k,\beta_k}^* \varphi = \frac{1}{\sqrt{b_{1,\beta_1}^{(m_1)}}} \cdots \frac{1}{\sqrt{b_{k,\beta_k}^{(m_k)}}} a_{\beta_1, \dots, \beta_k}.$$

On the other hand, according to Lemma 2.1, $(I - \Phi_{q_1, \mathbf{W}_1})^{m_1} \cdots (I - \Phi_{q_k, \mathbf{W}_k})^{m_k}(I) = \mathbf{P}_{\mathbb{C}}$, where $\mathbf{P}_{\mathbb{C}}$ is the orthogonal projection from $\otimes_{i=1}^k F^2(H_{n_i})$ onto $\mathbb{C}1 \subset \otimes_{i=1}^k F^2(H_{n_i})$. Hence, and using the fact that $\mathcal{M}$ is reducing for each $\mathbf{W}_{i,j}$, we deduce that $a_{\beta_1, \dots, \beta_k} \in \mathcal{M}$, so $1 \in \mathcal{M}$. Using again that $\mathcal{M}$ is invariant under the operators $\mathbf{W}_{i,j}$, we have $\mathcal{M} = \otimes_{i=1}^k F^2(H_{n_i})$. This completes the proof. $\square$

Let $\mathbf{T} = (T_1, \dots, T_k) \in \mathbf{D}_{\mathbf{f}}^{\mathbf{m}}(\mathcal{H})$ and $\mathbf{T}' = (T_1', \dots, T_k') \in \mathbf{D}_{\mathbf{f}}^{\mathbf{m}}(\mathcal{H}')$ be k-tuples with $T_i := (T_{i,1}, \dots, T_{i,n_i})$ and $T_i' := (T_{i,1}', \dots, T_{i,n_i}')$. We say that $\mathbf{T}$ is unitarily equivalent to $\mathbf{T}'$ if there is a unitary operator $U : \mathcal{H} \to \mathcal{H}'$ such that $T_{i,j} = U^* T_{i,j}' U$ for all $i \in \{1, \dots, k\}$ and $j \in \{1, \dots, n_i\}$.

Theorem 5.6. *Let $\mathbf{T} = (T_1, \dots, T_k)$ be a pure k-tuple in the noncommutative polydomain $\mathbf{D}_{\mathbf{f}}^{\mathbf{m}}(\mathcal{H})$ and let*

$$\mathbf{K}_{\mathbf{f},\mathbf{T}} : \mathcal{H} \to F^2(H_{n_1}) \otimes \cdots \otimes F^2(H_{n_k}) \otimes \overline{\mathbf{\Delta}_{\mathbf{f},\mathbf{T}}^{\mathbf{m}}(I)(\mathcal{H})}$$

be the noncommutative Berezin kernel. Then the subspace $\mathbf{K}_{\mathbf{f},\mathbf{T}}\mathcal{H}$ is co-invariant under each operator $\mathbf{W}_{i,j} \otimes I_{\overline{\mathbf{\Delta}_{\mathbf{f},\mathbf{T}}^{\mathbf{m}}\mathcal{H}}}$ for any $i \in \{1, \dots, k\}$, $j \in \{1, \dots, n_i\}$ and the dilation provided by the relation

$$\mathbf{T}_{(\alpha)} = \mathbf{K}_{\mathbf{f},\mathbf{T}}^* (\mathbf{W}_{(\alpha)} \otimes I_{\overline{\mathbf{\Delta}_{\mathbf{f},\mathbf{T}}^{\mathbf{m}}(I)(\mathcal{H})}}) \mathbf{K}_{\mathbf{f},\mathbf{T}}, \qquad (\alpha) \in \mathbb{F}_{n_1}^+ \times \cdots \times \mathbb{F}_{n_k}^+,$$

is minimal. If $\mathbf{f} = \mathbf{q} = (q_1, \ldots, q_k)$ *is a* k*-tuple of positive regular noncommutative polynomials and*

$$\overline{\text{span}}\,\{\mathbf{W}_{(\alpha)}\mathbf{W}^*_{(\beta)}) : \ (\alpha), (\beta) \in \mathbb{F}^+_{n_1} \times \cdots \times \mathbb{F}^+_{n_k}\} = C^*(\mathbf{W}_{i,j}),$$

then the minimal dilation of $\mathbf{T}$ *is unique up to an isomorphism.*

Proof. Due to Theorem 2.2, we have $\mathbf{K}_{\mathbf{f},\mathbf{T}} T^*_{i,j} = (\mathbf{W}^*_{i,j} \otimes I)\mathbf{K}_{\mathbf{f},\mathbf{T}}$ for any $i \in \{1, \ldots, k\}$ and $j \in \{1, \ldots, n_i\}$, where the noncommutative Berezin kernel $\mathbf{K}_{\mathbf{f},\mathbf{T}}$ is an isometry. On the other hand, the definition of the Berezin kernel $\mathbf{K}_{\mathbf{f},\mathbf{T}}$ implies

$$(\mathbf{P}_{\mathbb{C}} \otimes I_{\overline{\mathbf{\Delta}^{\mathbf{m}}_{\mathbf{f},\mathbf{T}}(I)(\mathcal{H})}})K_{\mathbf{f},\mathbf{T}}\mathcal{H} = \overline{\mathbf{\Delta}^{\mathbf{m}}_{\mathbf{f},\mathbf{T}}(I)(\mathcal{H})}.$$

Using Theorem 5.1 in the particular case when $\mathcal{M} := K_{\mathbf{f},\mathbf{T}}\mathcal{H}$ and $\mathcal{E} := \overline{\mathbf{\Delta}^{\mathbf{m}}_{\mathbf{f},\mathbf{T}}(I)(\mathcal{H})}$, we deduce that the subspace $\mathbf{K}_{\mathbf{f},\mathbf{T}}\mathcal{H}$ is cyclic for $\mathbf{W}_{i,j} \otimes I_{\mathcal{E}}$ for $i \in \{1, \ldots, k\}$ and $j \in \{1, \ldots, n_i\}$, which proves the minimality of the dilation, i.e.,

$$(\otimes^k_{i=1} F^2(H_{n_i})) \otimes \overline{\mathbf{\Delta}^{\mathbf{m}}_{\mathbf{f},\mathbf{T}}(I)(\mathcal{H})} = \bigvee_{(\alpha) \in \mathbb{F}^+_{n_1} \times \cdots \times \mathbb{F}^+_{n_k}} (\mathbf{W}_{(\alpha)} \otimes I_{\overline{\mathbf{\Delta}^{\mathbf{m}}_{\mathbf{f},\mathbf{T}}(I)(\mathcal{H})}})\mathbf{K}_{\mathbf{f},\mathbf{T}}\mathcal{H}. \tag{5.5}$$

To prove the last part of the theorem, assume that $\mathbf{f} = \mathbf{q} = (q_1, \ldots, q_k)$ is a k-tuple of positive regular noncommutative polynomials and that the relation in the theorem holds. Consider another minimal dilation of $\mathbf{T}$, i.e.,

$$\mathbf{T}_{(\alpha)} = V^*(\mathbf{W}_{(\alpha)} \otimes I_{\mathcal{D}})V, \qquad (\alpha) \in \mathbb{F}^+_{n_1} \times \cdots \times \mathbb{F}^+_{n_k}, \tag{5.6}$$

where $V : \mathcal{H} \to (\otimes^k_{i=1} F^2(H_{n_i})) \otimes \mathcal{D}$ is an isometry, $V\mathcal{H}$ is co-invariant under each operator $\mathbf{W}_{i,j} \otimes I_{\mathcal{D}}$, and

$$(\otimes^k_{i=1} F^2(H_{n_i})) \otimes \mathcal{D} = \bigvee_{(\alpha) \in \mathbb{F}^+_{n_1} \times \cdots \times \mathbb{F}^+_{n_k}} (\mathbf{W}_{(\alpha)} \otimes I_{\mathcal{D}})V\mathcal{H}. \tag{5.7}$$

Due to Theorem 2.4, there exists a unique unital completely positive linear map $\mathbf{\Psi}_{\mathbf{q},\mathbf{T}} : C^*(\mathbf{W}_{i,j}) \to B(\mathcal{H})$ such that

$$\mathbf{\Psi}_{\mathbf{q},\mathbf{T}}\left(\sum_{\gamma=1}^{s} p_\gamma(\mathbf{W}_{i,j})q_\gamma(\mathbf{W}_{i,j})^*\right) = \sum_{\gamma=1}^{s} p_\gamma(T_{i,j})q_\gamma(T_{i,j})^*$$

for any $p_\gamma(\mathbf{W}_{i,j}), q_\gamma(\mathbf{W}_{i,j}) \in \mathcal{P}(\mathbf{W})$ and $s \in \mathbb{N}$. Consider the $*$-representations

$$\pi_1 : C^*(\mathbf{W}_{i,j}) \to B((\otimes^k_{i=1} F^2(H_{n_i})) \otimes \overline{\mathbf{\Delta}^{\mathbf{m}}_{\mathbf{q},\mathbf{T}}(I)(\mathcal{H})}), \quad \pi_1(X) := X \otimes I_{\overline{\mathbf{\Delta}^{\mathbf{m}}_{\mathbf{q},\mathbf{T}}(I)(\mathcal{H})}}$$

and

$$\pi_2 : C^*(\mathbf{W}_{i,j}) \to B((\otimes^k_{i=1} F^2(H_{n_i})) \otimes \mathcal{D}), \quad \pi_2(X) := X \otimes I_{\mathcal{D}}.$$

Since the subspaces $\mathbf{K}_{\mathbf{q},\mathbf{T}}\mathcal{H}$ and $V\mathcal{H}$ are co-invariant for each operator $\mathbf{W}_{i,j} \otimes I_{\overline{\mathbf{\Delta}^{\mathbf{m}}_{\mathbf{q},\mathbf{T}}(I)\mathcal{H}}}$, the relation (5.6) implies

$$\mathbf{\Psi}_{\mathbf{q},\mathbf{T}}(X) = \mathbf{K}^*_{\mathbf{q},\mathbf{T}}\pi_1(X)\mathbf{K}_{\mathbf{q},\mathbf{T}} = V^*\pi_2(X)V, \qquad X \in C^*(\mathbf{W}_{i,j}).$$

Due to relations (5.5) and (5.7), we deduce that π_1 and π_2 are minimal Stinespring dilations of the completely positive linear map $\mathbf{\Psi}_{\mathbf{q},\mathbf{T}}$. Since these representations are unique up to an isomorphism, there exists a unitary operator $U : (\otimes^k_{i=1} F^2(H_{n_i})) \otimes \overline{\mathbf{\Delta}^{\mathbf{m}}_{\mathbf{q},\mathbf{T}}(I)(\mathcal{H})} \to (\otimes^k_{i=1} F^2(H_{n_i})) \otimes \mathcal{D}$ such that

$$U(\mathbf{W}_{i,j} \otimes I_{\overline{\mathbf{\Delta}^{\mathbf{m}}_{\mathbf{q},\mathbf{T}}(I)(\mathcal{H})}}) = (\mathbf{W}_{i,j} \otimes I_{\mathcal{D}})U$$

and $U\mathbf{K}_{\mathbf{q},\mathbf{T}} = V$. Taking into account that U is unitary, we deduce that

$$U(\mathbf{W}^*_{i,j} \otimes I_{\overline{\mathbf{\Delta}^{\mathbf{m}}_{\mathbf{q},\mathbf{T}}(I)(\mathcal{H})}}) = (\mathbf{W}^*_{i,j} \otimes I_{\mathcal{D}})U.$$

Since $C^*(\mathbf{W}_{i,j})$ is irreducible (see Lemma 5.5), we must have $U = I \otimes Z$, where $Z \in B(\overline{\mathbf{\Delta}^{\mathbf{m}}_{\mathbf{q},\mathbf{T}}(\mathcal{H})}, \mathcal{D})$ is a unitary operator. This implies that $\dim \overline{\mathbf{\Delta}^{\mathbf{m}}_{\mathbf{q},\mathbf{T}}(\mathcal{H})} = \dim \mathcal{D}$ and $U\mathbf{K}_{\mathbf{q},\mathbf{T}}\mathcal{H} = V\mathcal{H}$, which proves that the two dilations are unitarily equivalent. The proof is complete. $\square$

Let $\mathcal{D}$ be a Hilbert space such that the Hilbert space $\mathcal{H}$ can be identified with a co-invariant subspace of $(\otimes_{i=1}^k F^2(H_{n_i}))\otimes\mathcal{D}$ under each operator $\mathbf{W}_{i,j}\otimes I_\mathcal{D}$ for any $i\in\{1,\dots,k\}$, $j\in\{1,\dots,n_i\}$ and such that $\mathbf{T}_{(\alpha)}=V^*(\mathbf{W}_{(\alpha)}\otimes I_\mathcal{D})V$ for $(\alpha)\in\mathbb{F}_{n_1}^+\times\cdots\times\mathbb{F}_{n_k}^+$. The dilation index of $\mathbf{T}$ is the minimum dimension of $\mathcal{D}$ with the above mentioned property. We remark that the dilation index of $\mathbf{T}$ coincides with $\operatorname{rank}\boldsymbol{\Delta}_{\mathbf{f},\mathbf{T}}^{\mathbf{m}}(I)$. Indeed, since $\boldsymbol{\Delta}_{\mathbf{f},\mathbf{W}}^{\mathbf{m}}(I)=\mathbf{P}_\mathbb{C}$, where $\mathbf{P}_\mathbb{C}$ is the orthogonal projection from $\otimes_{i=1}^k F^2(H_{n_i})$ onto $\mathbb{C}1\subset\otimes_{i=1}^k F^2(H_{n_i})$, we deduce that $\boldsymbol{\Delta}_{\mathbf{f},\mathbf{T}}^{\mathbf{m}}(I)=P_\mathcal{H}\left[\mathbf{P}_\mathbb{C}\otimes I_\mathcal{D}\right]|\mathcal{H}$. Hence, $\operatorname{rank}\boldsymbol{\Delta}_{\mathbf{f},\mathbf{T}}^{\mathbf{m}}(I)\leq\dim\mathcal{D}$. Now, Theorem 5.6 implies that the dilation index of T is equal to $\operatorname{rank}\boldsymbol{\Delta}_{\mathbf{f},\mathbf{T}}^{\mathbf{m}}(I)$.

Proposition 5.7. *Let $\mathbf{q}=(q_1,\dots,q_k)$ is a k-tuple of positive regular noncommutative polynomials such that*

$$\overline{\operatorname{span}}\,\{\mathbf{W}_{(\alpha)}\mathbf{W}_{(\beta)}^*):\ (\alpha),(\beta)\in\mathbb{F}_{n_1}^+\times\cdots\times\mathbb{F}_{n_k}^+\}=C^*(\mathbf{W}_{i,j}).$$

A pure k-tuple $\mathbf{T}=(T_1,\dots,T_k)\in\mathbf{D}_\mathbf{q}^\mathbf{m}(\mathcal{H})$ has $\operatorname{rank}\boldsymbol{\Delta}_{\mathbf{q},\mathbf{T}}^{\mathbf{m}}(I)=n$, $n=1,2,\dots,\infty$, if and only if it is unitarily equivalent to one obtained by compressing $(\mathbf{W}_1\otimes I_{\mathbb{C}^n},\dots,\mathbf{W}_k\otimes I_{\mathbb{C}^n})$ to a co-invariant subspace $\mathcal{M}\subset(\otimes_{i=1}^k F^2(H_{n_i}))\otimes\mathbb{C}^n$ under each operator $\mathbf{W}_{i,j}\otimes I_{\mathbb{C}^n}$, $i\in\{1,\dots,k\}$ and $j\in\{1,\dots,n_i\}$, with the property that $\dim[(\mathbf{P}_\mathbb{C}\otimes I_{\mathbb{C}^n})\mathcal{M}]=n$, where $\mathbf{P}_\mathbb{C}$ is the orthogonal projection from $\otimes_{i=1}^k F^2(H_{n_i})$ onto $\mathbb{C}1$.

Proof. The direct implication is a consequence of Theorem 5.6. To prove the converse, assume that

$$\mathbf{T}_{(\alpha)}=P_\mathcal{H}(\mathbf{W}_{(\alpha)}\otimes I_{\mathbb{C}^n})|_\mathcal{H},\qquad(\alpha)\in\mathbb{F}_{n_1}^+\times\cdots\times\mathbb{F}_{n_k}^+$$

where $\mathcal{H}\subset(\otimes_{i=1}^k F^2(H_{n_i}))\otimes\mathbb{C}^n$ is a co-invariant subspace under each operator $\mathbf{W}_{i,j}\otimes I_{\mathbb{C}^n}$ for any $i\in\{1,\dots,k\}$, $j\in\{1,\dots,n_i\}$, such that $\dim(\mathbf{P}_\mathbb{C}\otimes I_{\mathbb{C}^n})\mathcal{H}=n$. Note that $\mathbf{T}$ is a pure element in the noncommutative polydomain $\mathbf{D}_\mathbf{q}^\mathbf{m}(\mathcal{H})$. First, we consider the case when $n<\infty$. Since $(\mathbf{P}_\mathbb{C}\otimes I_{\mathbb{C}^n})\mathcal{H}\subseteq\mathbb{C}^n$ and $\dim(\mathbf{P}_\mathbb{C}\otimes I_{\mathbb{C}^n})\mathcal{H}=n$, we deduce that $(\mathbf{P}_\mathbb{C}\otimes I_{\mathbb{C}^n})\mathcal{H}=\mathbb{C}^n$. This condition is equivalent to the equality $\mathcal{H}^\perp\cap\mathbb{C}^n=\{0\}$. Since $\boldsymbol{\Delta}_{\mathbf{q},\mathbf{W}}^{\mathbf{m}}(I)=\mathbf{P}_\mathbb{C}$, where $\mathbf{P}_\mathbb{C}$ is the orthogonal projection from $\otimes_{i=1}^k F^2(H_{n_i})$ onto $\mathbb{C}1\subset\otimes_{i=1}^k F^2(H_{n_i})$, we deduce that $\boldsymbol{\Delta}_{\mathbf{q},\mathbf{T}}^{\mathbf{m}}(I)=P_\mathcal{H}\left[\mathbf{P}_\mathbb{C}\otimes I_{\mathbb{C}^n}\right]|_\mathcal{H}=P_\mathcal{H}\mathbb{C}^n$. Consequently, we have $\operatorname{rank}\boldsymbol{\Delta}_{\mathbf{q},\mathbf{T}}^{\mathbf{m}}(I)=\dim P_\mathcal{H}\mathbb{C}^n$. If we assume that $\operatorname{rank}\boldsymbol{\Delta}_{\mathbf{q},\mathbf{T}}^{\mathbf{m}}(I)<n$, then there exists $h\in\mathbb{C}^n$, $h\neq0$, with $P_\mathcal{H}h=0$. This contradicts the fact that $\mathcal{H}^\perp\cap\mathbb{C}^n=\{0\}$. Therefore, we must have $\operatorname{rank}\boldsymbol{\Delta}_{\mathbf{q},\mathbf{T}}^{\mathbf{m}}(I)=n$.

Now, we consider the case when $n=\infty$. According to Theorem 5.1 and its proof, we have

$$(\otimes_{i=1}^k F^2(H_{n_i}))\otimes\mathcal{E}=\bigvee_{(\alpha)\in\mathbb{F}_{n_1}^+\times\cdots\times\mathbb{F}_{n_k}^+}(\mathbf{W}_{(\alpha)}\otimes I_{\mathbb{C}^n})\mathcal{H}$$

where $\mathcal{E}:=(\mathbf{P}_\mathbb{C}\otimes I_{\mathbb{C}^n})\mathcal{H}$. Since $(\otimes_{i=1}^k F^2(H_{n_i}))\otimes\mathcal{E}$ is reducing for each operator $\mathbf{W}_{i,j}\otimes I_{\mathbb{C}^m}$, we deduce that $\mathbf{T}_{(\alpha)}=P_\mathcal{H}(\mathbf{W}_{(\alpha)}\otimes I_\mathcal{E})|_\mathcal{H}$, $(\alpha)\in\mathbb{F}_{n_1}^+\times\cdots\times\mathbb{F}_{n_k}^+$. The uniqueness of the minimal dilation of $\mathbf{T}$ (see Theorem 5.6) implies $\dim\overline{\boldsymbol{\Delta}_{\mathbf{q},\mathbf{T}}^{\mathbf{m}}(\mathbf{I})\mathcal{H}}=\dim\mathcal{E}=\infty$. This completes the proof. $\square$

We can characterize now the pure n-tuples of operators in the noncommutative polydomain $\mathbf{D}_\mathbf{q}^\mathbf{m}(\mathcal{H})$, having rank one, i.e., $\operatorname{rank}\boldsymbol{\Delta}_{\mathbf{q},\mathbf{T}}^{\mathbf{m}}(I)=1$.

Corollary 5.8. *Under the hypothesis of Proposition 5.7, the following statements hold.*

(i) *If $\mathcal{M}\subset\otimes_{i=1}^k F^2(H_{n_i})$ is a co-invariant subspace under each operator $\mathbf{W}_{i,j}$, where $i\in\{1,\dots,k\}$ and $j\in\{1,\dots,n_i\}$, then*

$$\mathbf{T}:=(T_1,\dots,T_k),\quad T_i:=(P_\mathcal{M}\mathbf{W}_{i,1}|_\mathcal{M},\dots,P_\mathcal{M}\mathbf{W}_{i,n_i}|_\mathcal{M}),$$

is a pure k-tuple in $\mathbf{D}_\mathbf{q}^\mathbf{m}(\mathcal{M})$ such that $\operatorname{rank}\boldsymbol{\Delta}_{\mathbf{q},\mathbf{T}}^{\mathbf{m}}=1$.

(ii) *If $\mathcal{M}'$ is another co-invariant subspace under each operator $\mathbf{W}_{i,j}$, which gives rise to an k-tuple $\mathbf{T}'$, then $\mathbf{T}$ and $\mathbf{T}'$ are unitarily equivalent if and only if $\mathcal{M}=\mathcal{M}'$.*

Proof. Since $\boldsymbol{\Delta}_{\mathbf{q},\mathbf{T}}^{\mathbf{m}}(I)=P_\mathcal{M}\mathbf{P}_\mathbb{C}|_\mathcal{M}$ we have $\operatorname{rank}\boldsymbol{\Delta}_{\mathbf{q},\mathbf{T}}^{\mathbf{m}}(I)\leq1$. On the other hand, it is clear that $\mathbf{T}$ is pure. This also implies that $\boldsymbol{\Delta}_{\mathbf{q},\mathbf{T}}^{\mathbf{m}}(I)\neq0$, so $\operatorname{rank}\boldsymbol{\Delta}_{\mathbf{q},\mathbf{T}}^{\mathbf{m}}(I)\geq1$. Therefore, $\operatorname{rank}\boldsymbol{\Delta}_{\mathbf{q},\mathbf{T}}^{\mathbf{m}}(I)=1$.

To prove (ii), note that, as in the proof of Theorem 5.6, one can show that $\mathbf{T}$ and $\mathbf{T}'$ are unitarily equivalent if and only if there exists a unitary operator $\Lambda:\otimes_{i=1}^k F^2(H_{n_i})\to\otimes_{i=1}^k F^2(H_{n_i})$ such that $\Lambda\mathbf{W}_{i,j}=\mathbf{W}_{i,j}\Lambda$, $i\in\{1,\dots,k\}$, $j\in\{1,\dots,n_i\}$, and $\Lambda\mathcal{M}=\mathcal{M}'$. Hence $\Lambda\mathbf{W}_{i,j}^*=\mathbf{W}_{i,j}^*\Lambda$. Since $C^*(\mathbf{W}_{i,j})$ is irreducible (see Theorem 7.1), Λ must be a scalar multiple of the identity. Therefore, we have $\mathcal{M}=\Lambda\mathcal{M}=\mathcal{M}'$. $\square$

6. Characteristic functions and operator models

We provide a characterization for the class of tuples of operators in $\mathbf{D_f^m}(\mathcal{H})$ which admit characteristic functions. We prove that the characteristic function is a complete unitary invariant for the class of completely non-coisometric tuples and provide an operator model for this class of elements in terms of their characteristic functions.

Let $\mathbf{W} := (\mathbf{W}_1, \ldots, \mathbf{W}_k)$ be the the universal model associated with the abstract noncommutative domain $\mathbf{D_f^m}$. We say that two multi-analytic operator $\Phi : (\otimes_{i=1}^k F^2(H_{n_i}))\otimes\mathcal{K}_1 \to (\otimes_{i=1}^k F^2(H_{n_i}))\otimes\mathcal{K}_2$ and $\Phi' : (\otimes_{i=1}^k F^2(H_{n_i}))\otimes\mathcal{K}_1' \to (\otimes_{i=1}^k F^2(H_{n_i}))\otimes\mathcal{K}_2'$ coincide if there are two unitary operators $\tau_j \in B(\mathcal{K}_j, \mathcal{K}_j')$ such that

$$\Phi'(I_{\otimes_{i=1}^k F^2(H_{n_i})} \otimes \tau_1) = (I_{\otimes_{i=1}^k F^2(H_{n_i})} \otimes \tau_2)\Phi.$$

Lemma 6.1. *Let $\Phi_s : (\otimes_{i=1}^k F^2(H_{n_i})) \otimes \mathcal{H}_s \to (\otimes_{i=1}^k F^2(H_{n_i})) \otimes \mathcal{K},\ \ s = 1, 2$, be multi-analytic operators with respect to $\mathbf{W} := (\mathbf{W}_1, \ldots, \mathbf{W}_k)$ such that $\Phi_1\Phi_1^* = \Phi_2\Phi_2^*$. Then there is a unique partial isometry $V : \mathcal{H}_1 \to \mathcal{H}_2$ such that*

$$\Phi_1 = \Phi_2(I_{\otimes_{i=1}^k F^2(H_{n_i})} \otimes V),$$

where $(I_{\otimes_{i=1}^k F^2(H_{n_i})} \otimes V)$ is an inner multi-analytic operator with initial space $\operatorname{supp}(\Phi_1)$ *and final space* $\operatorname{supp}(\Phi_2)$*. In particular, the multi-analytic operators $\Phi_1|_{\operatorname{supp}(\Phi_1)}$ and $\Phi_2|_{\operatorname{supp}(\Phi_2)}$ coincide.*

Proof. Due to Lemma 2.1, $(id - \Phi_{f_1,\mathbf{W}_1})^{m_1} \cdots (id - \Phi_{f_k,\mathbf{W}_k})^{m_k}(I) = \mathbf{P}_\mathbb{C}$, where $\mathbf{P}_\mathbb{C}$ is the orthogonal projection from $\otimes_{i=1}^k F^2(H_{n_i})$ onto $\mathbb{C}1 \subset \otimes_{i=1}^k F^2(H_{n_i})$. Since Φ_1, Ψ_2 are multi-analytic operators with respect to $\mathbf{W}$, we deduce that $\Phi_1(\mathbf{P}_\mathbb{C} \otimes I_{\mathcal{H}_1})\Phi_1^* = \Phi_2(\mathbf{P}_\mathbb{C} \otimes I_{\mathcal{H}_2})\Phi_2^*$. Consequently, we have

$$\|(\mathbf{P}_\mathbb{C} \otimes I_{\mathcal{H}_1})\Phi_1^* x\| = \|(\mathbf{P}_\mathbb{C} \otimes I_{\mathcal{H}_2})\Phi_2^* x\|, \qquad x \in (\otimes_{i=1}^k F^2(H_{n_i})) \otimes \mathcal{K}.$$

Set $\mathcal{L}_s := (\mathbf{P}_\mathbb{C} \otimes I_{\mathcal{H}_s})\overline{\Phi_s^*((\otimes_{i=1}^k F^2(H_{n_i})) \otimes \mathcal{K})}$, $s = 1, 2$, and define the unitary operator $U : \mathcal{L}_1 \to \mathcal{L}_2$ by

$$U(\mathbf{P}_\mathbb{C} \otimes I_{\mathcal{H}_1})\Phi_1^* x := (\mathbf{P}_\mathbb{C} \otimes I_{\mathcal{H}_2})\Phi_2^* x, \qquad x \in (\otimes_{i=1}^k F^2(H_{n_i})) \otimes \mathcal{K}.$$

This implies that there is a unique partial isometry $V : \mathcal{H}_1 \to \mathcal{H}_2$ with initial space $\mathcal{L}_1$ and final space $\mathcal{L}_2$, extending U. Moreover, we have $\Phi_1 V^* = \Phi_2|_{1\otimes\mathcal{H}_2}$. Since Φ_1, Ψ_2 are multi-analytic operators with respect to $\mathbf{W}$, we deduce that $\Phi_1(I_{\otimes_{i=1}^k F^2(H_{n_i})} \otimes V^*) = \Phi_2$. Hence, the result follows. Now, the last part of the lemma is clear. $\square$

We say that $\mathbf{T} = (T_1, \ldots, T_k) \in \mathbf{D_f^m}(\mathcal{H})$ has characteristic function if there is a Hilbert space $\mathcal{E}$ and a multi-analytic operator $\Psi : (\otimes_{i=1}^k F^2(H_{n_i})) \otimes \mathcal{E} \to (\otimes_{i=1}^k F^2(H_{n_i})) \otimes \overline{\mathbf{\Delta_{f,T}^m}(I)(\mathcal{H})}$ with respect to $W_{i,j}$, $i \in \{1, \ldots, k\}$, $j \in \{1, \ldots, n_i\}$, such that

$$\mathbf{K_{f,T}}\mathbf{K_{f,T}^*} + \Psi\Psi^* = I.$$

According to Lemma 6.1, if there is a characteristic function for $\mathbf{T} \in \mathbf{D_f^m}(\mathcal{H})$, then it is essentially unique.

We give now an example of a class of elements $\mathbf{T} \in \mathbf{D_f^m}(\mathcal{H})$ which have characteristic function. Let $\Psi : (\otimes_{i=1}^k F^2(H_{n_i})) \otimes \mathcal{E} \to (\otimes_{i=1}^k F^2(H_{n_i})) \otimes \mathcal{G}$ be an inner multi-analytic operator with $\Psi(0) = 0$ and consider the subspace $\mathcal{M} := \Psi((\otimes_{i=1}^k F^2(H_{n_i})) \otimes \mathcal{E})$. Note that $\mathcal{M}$ is invariant under each operator $\mathbf{W}_{i,j}$ and define $T_{i,j} := P_{\mathcal{M}^\perp}(\mathbf{W}_{i,j} \otimes I_\mathcal{G})|_{\mathcal{M}^\perp}$ for $i \in \{1, \ldots, k\}$ and $j \in \{1, \ldots, n_i\}$. Set $\mathbf{T} := (T_1, \ldots, T_k)$, where $T_i = (T_{i,1}, \ldots, T_{i,j})$, and note that

$$\mathbf{\Delta_{f,T}^m}(I_{\mathcal{M}^\perp}) = P_{\mathcal{M}^\perp}\mathbf{\Delta^m_{f,W\otimes I_\mathcal{G}}}(I_\mathcal{G})|_{\mathcal{M}^\perp} = P_{\mathcal{M}^\perp}(\mathbf{P}_\mathbb{C} \otimes I_\mathcal{G})|_{\mathcal{M}^\perp}.$$

Since $\Psi(0) = 0$, we have $1 \otimes \mathcal{G} \subset \mathcal{M}^\perp$ and, consequently, $\mathbf{\Delta_{f,T}^m}(I_{\mathcal{M}^\perp})^{1/2} = (\mathbf{P}_\mathbb{C} \otimes I_\mathcal{G})|_{\mathcal{M}^\perp}$. Consider an arbitrary vector

$$h = \sum_{\beta_i \in \mathbb{F}_{n_i}^+, i=1,\ldots,k} e^1_{\beta_1} \otimes \cdots \otimes e^k_{\beta_k} \otimes h_{\beta_1,\ldots,\beta_k}$$

in $\mathcal{M}^\perp \subset (\otimes_{i=1}^k F^2(H_{n_i}))\otimes \mathcal{G}$. Using the definition of the noncommutative Berezin kernel and relation (2.6), we obtain

$$\begin{aligned}\mathbf{K}_{\mathbf{f},\mathbf{T}}h &:= \sum_{\beta_i\in\mathbb{F}_{n_i}^+, i=1,\ldots,k} \sqrt{b_{1,\beta_1}^{(m_1)}}\cdots\sqrt{b_{k,\beta_k}^{(m_k)}}e_{\beta_1}^1\otimes\cdots\otimes e_{\beta_k}^k\otimes(\mathbf{P}_\mathbb{C}\otimes I_\mathcal{G})(\mathbf{W}_{1,\beta_1}^*\cdots\mathbf{W}_{k,\beta_k}^*\otimes I_\mathcal{G})^*h\\ &= \sum_{\beta_i\in\mathbb{F}_{n_i}^+, i=1,\ldots,k} \sqrt{b_{1,\beta_1}^{(m_1)}}\cdots\sqrt{b_{k,\beta_k}^{(m_k)}}e_{\beta_1}^1\otimes\cdots\otimes e_{\beta_k}^k\otimes\frac{1}{\sqrt{b_{1,\beta_1}^{(m_1)}}}\cdots\frac{1}{\sqrt{b_{k,\beta_k}^{(m_k)}}}(1\otimes h_{\beta_1,\ldots,\beta_k}) = h\end{aligned}$$

Consequently, $\mathbf{K}_{\mathbf{f},\mathbf{T}}$ can be identified with the injection of $\mathcal{M}^\perp$ into $(\otimes_{i=1}^k F^2(H_{n_i}))\otimes\mathcal{G}$, and $\mathbf{K}_{\mathbf{f},\mathbf{T}}\mathbf{K}_{\mathbf{f},\mathbf{T}}^*$ can be identified with the orthogonal projection $P_{\mathcal{M}^\perp}$. Therefore, $\mathbf{K}_{\mathbf{f},\mathbf{T}}\mathbf{K}_{\mathbf{f},\mathbf{T}}^* + \Psi\Psi^* = I$, which proves our assertion.

We also remark that in the particular case when $k=1$ and $m_1=1$, all the elements in the noncommutative domain $\mathbf{D}_{f_1}^1$ have characteristic functions.

Theorem 6.2. *A k-tuple $\mathbf{T}=(T_1,\ldots,T_k)$ in the noncommutative polydomain $\mathbf{D}_\mathbf{f}^\mathbf{m}(\mathcal{H})$ admits a characteristic function if and only if*

$$\mathbf{\Delta}_{\mathbf{f},\mathbf{W}\otimes I}^\mathbf{p}(I-\mathbf{K}_{\mathbf{f},\mathbf{T}}\mathbf{K}_{\mathbf{f},\mathbf{T}}^*)\geq 0$$

for any $\mathbf{p}:=(p_1,\ldots,p_k)\in\mathbb{Z}_+^k$ such that $\mathbf{p}\leq\mathbf{m}$, where $\mathbf{K}_{\mathbf{f},\mathbf{T}}$ is the noncommutative Berezin kernel associated with $\mathbf{T}$.

Proof. If $\mathbf{T}$ has characteristic function, then there is a multi-analytic operator Ψ with the property that $\mathbf{K}_{\mathbf{f},\mathbf{T}}\mathbf{K}_{\mathbf{f},\mathbf{T}}^*+\Psi\Psi^*=I$. Using the multi-analyticity of Ψ, we have

$$\mathbf{\Delta}_{\mathbf{f},\mathbf{W}\otimes I}^\mathbf{p}(I-\mathbf{K}_{\mathbf{f},\mathbf{T}}\mathbf{K}_{\mathbf{f},\mathbf{T}}^*) = \Psi\mathbf{\Delta}_{\mathbf{f},\mathbf{W}\otimes I}^\mathbf{p}(I)\Psi^*\geq 0,$$

for any $\mathbf{p}:=(p_1,\ldots,p_k)\in\mathbb{Z}_+^k$ such that $\mathbf{p}\leq\mathbf{m}$. For the converse, we apply Theorem 5.2 to the operator $Y=I-\mathbf{K}_{\mathbf{f},\mathbf{T}}\mathbf{K}_{\mathbf{f},\mathbf{T}}^*$ and complete the proof. $\square$

If $\mathbf{T}$ has characteristic function, the multi-analytic operator M provided by the proof of Theorem 5.2 when $Y=I-\mathbf{K}_{\mathbf{f},\mathbf{T}}\mathbf{K}_{\mathbf{f},\mathbf{T}}^*$, which we denote by $\Theta_{\mathbf{f},\mathbf{T}}$, is called the *characteristic function* of $\mathbf{T}$. More precisely, $\Theta_{\mathbf{f},\mathbf{T}}$ is the multi-analytic operator

$$\Theta_{\mathbf{f},\mathbf{T}}:(\otimes_{i=1}^k F^2(H_{n_i}))\otimes\overline{\mathbf{\Delta}_{\mathbf{f},\mathbf{M}_\mathbf{T}}^\mathbf{m}(I)(\mathcal{M}_T)}\to(\otimes_{i=1}^k F^2(H_{n_i}))\otimes\overline{\mathbf{\Delta}_{\mathbf{f},\mathbf{T}}^\mathbf{m}(I)(\mathcal{H})}$$

defined by $\Theta_{\mathbf{f},\mathbf{T}}:=(I-\mathbf{K}_{\mathbf{f},\mathbf{T}}\mathbf{K}_{\mathbf{f},\mathbf{T}}^*)^{1/2}\mathbf{K}_{\mathbf{f},\mathbf{M}_\mathbf{T}}^*$, where

$$\mathbf{K}_{\mathbf{f},\mathbf{T}}:\mathcal{H}\to F^2(H_{n_1})\otimes\cdots\otimes F^2(H_{n_k})\otimes\overline{\mathbf{\Delta}_{\mathbf{f},\mathbf{T}}^\mathbf{m}(I)(\mathcal{H})}$$

is the noncommutative Berezin kernel associated with $\mathbf{T}$ and

$$\mathbf{K}_{\mathbf{f},\mathbf{M}_\mathbf{T}}:\mathcal{H}\to F^2(H_{n_1})\otimes\cdots\otimes F^2(H_{n_k})\otimes\overline{\mathbf{\Delta}_{\mathbf{f},\mathbf{M}_\mathbf{T}}^\mathbf{m}(I)(\mathcal{M}_\mathbf{T})}$$

is the noncommutative Berezin kernel associated with $\mathbf{M}_\mathbf{T}\in\mathbf{D}_\mathbf{f}^\mathbf{m}(\mathcal{M}_\mathbf{T})$. Here, we have

$$\mathcal{M}_\mathbf{T}:=\overline{\operatorname{range}(I-\mathbf{K}_{\mathbf{f},\mathbf{T}}\mathbf{K}_{\mathbf{f},\mathbf{T}}^*)}$$

and $\mathbf{M}_\mathbf{T}:=(M_1,\ldots,M_k)$ is the k-tuple with $M_i:=(M_{i,1},\ldots,M_{i,n_i})$ and $M_{i,j}\in B(\mathcal{M}_\mathbf{T})$ given by $M_{i,j}:=A_{i,j}^*$, where $A_{i,j}\in B(\mathcal{M}_\mathbf{T})$ is uniquely defined by

$$A_{i,j}\left[(I-\mathbf{K}_{\mathbf{f},\mathbf{T}}\mathbf{K}_{\mathbf{f},\mathbf{T}}^*)^{1/2}x\right]:=(I-\mathbf{K}_{\mathbf{f},\mathbf{T}}\mathbf{K}_{\mathbf{f},\mathbf{T}}^*)^{1/2}(\mathbf{W}_{i,j}\otimes I)x$$

for any $x\in(\otimes_{i=1}^k F^2(H_{n_i}))\otimes\overline{\mathbf{\Delta}_{\mathbf{f},\mathbf{T}}^\mathbf{m}(I)(\mathcal{H})}$. According to Theorem 5.2, we have $\mathbf{K}_{\mathbf{f},\mathbf{T}}\mathbf{K}_{\mathbf{f},\mathbf{T}}^*+\Theta_{\mathbf{f},\mathbf{T}}\Theta_{\mathbf{f},\mathbf{T}}^*=I$.

We denote by $\mathcal{C}_\mathbf{f}^\mathbf{m}(\mathcal{H})$ the set of all $\mathbf{T}=(T_1,\ldots,T_k)\in\mathbf{D}_\mathbf{f}^\mathbf{m}(\mathcal{H})$ which admit characteristic functions.

Theorem 6.3. *Let $\mathbf{T}=(T_1,\ldots,T_k)$ be a k-tuple in $\mathcal{C}_\mathbf{f}^\mathbf{m}(\mathcal{H})$. Then $\mathbf{T}$ is pure if and only if the characteristic function $\Theta_{\mathbf{f},\mathbf{T}}$ is an inner multi-analytic operator. Moreover, in this case $\mathbf{T}=(T_1,\ldots,T_k)$ is unitarily equivalent to $\mathbf{G}=(G_1,\ldots,G_k)$, where $G_i:=(G_{i,1},\ldots,G_{i,n_i})$ is defined by*

$$G_{i,j}:=P_{\mathbf{H}_{\mathbf{f},\mathbf{T}}}(\mathbf{W}_{i,j}\otimes I)|_{\mathbf{H}_{\mathbf{f},\mathbf{T}}}$$

and $P_{\mathbf{H}_{\mathbf{f},\mathbf{T}}}$ is the orthogonal projection of $(\otimes_{i=1}^k F^2(H_{n_i}))\otimes \overline{\mathbf{\Delta}^{\mathbf{m}}_{\mathbf{f},\mathbf{T}}(I)(\mathcal{H})}$ onto

$$\mathbf{H}_{\mathbf{f},\mathbf{T}}:=\left\{(\otimes_{i=1}^k F^2(H_{n_i}))\otimes \overline{\mathbf{\Delta}^{\mathbf{m}}_{\mathbf{f},\mathbf{T}}(I)(\mathcal{H})}\right\}\ominus \operatorname{range}\Theta_{\mathbf{f},\mathbf{T}}.$$

Proof. Assume that $\mathbf{T}$ is a pure k-tuple in $\mathcal{C}^{\mathbf{m}}_{\mathbf{f}}(\mathcal{H})$. Theorem 2.2 shows that the noncommutative Berezin kernel associated with $\mathbf{T}$, i.e.,

$$\mathbf{K}_{\mathbf{f},\mathbf{T}}:\mathcal{H}\to F^2(H_{n_1})\otimes\cdots\otimes F^2(H_{n_k})\otimes \overline{\mathbf{\Delta}^{\mathbf{m}}_{\mathbf{f},\mathbf{T}}(I)(\mathcal{H})}$$

is an isometry, the subspace $\mathbf{K}_{\mathbf{f},\mathbf{T}}\mathcal{H}$ is coinvariant under the operators $\mathbf{W}_{i,j}\otimes I_{\overline{\mathbf{\Delta}^{\mathbf{m}}_{\mathbf{f},\mathbf{T}}(I)(\mathcal{H})}}$, $i\in\{1,\ldots,k\}$, $j\in\{1,\ldots,n_i\}$, and $T_{i,j}=\mathbf{K}^*_{\mathbf{f},\mathbf{T}}(\mathbf{W}_{i,j}\otimes I_{\overline{\mathbf{\Delta}^{\mathbf{m}}_{\mathbf{f},\mathbf{T}}(I)(\mathcal{H})}})\mathbf{K}_{\mathbf{f},\mathbf{T}}$. Since $\mathbf{K}_{\mathbf{f},\mathbf{T}}\mathbf{K}^*_{\mathbf{f},\mathbf{T}}$ is the orthogonal projection of $(\otimes_{i=1}^k F^2(H_{n_i}))\otimes \overline{\mathbf{\Delta}^{\mathbf{m}}_{\mathbf{f},\mathbf{T}}(I)(\mathcal{H})}$ onto $\mathbf{K}_{\mathbf{f},\mathbf{T}}\mathcal{H}$ and $\mathbf{K}_{\mathbf{f},\mathbf{T}}\mathbf{K}^*_{\mathbf{f},\mathbf{T}}+\Theta_{\mathbf{f},\mathbf{T}}\Theta^*_{\mathbf{f},\mathbf{T}}=I$, we deduce that $\Theta_{\mathbf{f},\mathbf{T}}$ is a partial isometry and $\mathbf{K}_{\mathbf{f},\mathbf{T}}\mathcal{H}=\mathbf{H}_{\mathbf{f},\mathbf{T}}$. Since $\mathbf{K}_{\mathbf{f},\mathbf{T}}$ is an isometry, we can identify $\mathcal{H}$ with $\mathbf{K}_{\mathbf{f},\mathbf{T}}\mathcal{H}$. Therefore, $\mathbf{T}=(T_1,\ldots,T_k)$ is unitarily equivalent to $\mathbf{G}=(G_1,\ldots,G_k)$.

Conversely, if we assume that $\Theta_{\mathbf{f},\mathbf{T}}$ is inner, then it is a partial isometry. Due to the fact that $\mathbf{K}_{\mathbf{f},\mathbf{T}}\mathbf{K}^*_{\mathbf{f},\mathbf{T}}+\Theta_{\mathbf{f},\mathbf{T}}\Theta^*_{\mathbf{f},\mathbf{T}}=I$, the noncommutative Berezin kernel $\mathbf{K}_{\mathbf{f},\mathbf{T}}$ is a partial isometry. On the other hand, since $\mathbf{T}$ is completely non-coisometric, $\mathbf{K}_{\mathbf{f},\mathbf{T}}$ is a one-to-one partial isometry and, therefore, isometry. Due to Theorem 2.2, we have

$$\mathbf{K}^*_{\mathbf{f},\mathbf{T}}\mathbf{K}_{\mathbf{f},\mathbf{T}}=\lim_{\mathbf{q}=(q_1,\ldots,q_k)\in\mathbb{Z}_+^k}(id-\Phi^{q_k}_{f_k,T_k})\cdots(id-\Phi^{q_1}_{f_1,T_1})(I)=I$$

Consequently, $\mathbf{T}$ is a pure k-tuple. The proof is complete. □

Now, we are able to provide a model theorem for class of the completely non-coisometric k-tuple of operators in $\mathcal{C}^{\mathbf{m}}_{\mathbf{f}}(\mathcal{H})$.

Theorem 6.4. *Let $\mathbf{T}=(T_1,\ldots,T_k)$ be a completely non-coisometric k-tuple in $\mathcal{C}^{\mathbf{m}}_{\mathbf{f}}(\mathcal{H})$ and let $\mathbf{W}:=(\mathbf{W}_1,\ldots,\mathbf{W}_k)$ be the universal model associated to the abstract noncommutative domain $\mathbf{D}^{\mathbf{m}}_{\mathbf{f}}$. Set*

$$\mathcal{D}:=\overline{\mathbf{\Delta}^{\mathbf{m}}_{\mathbf{f},\mathbf{T}}(I)(\mathcal{H})},\qquad \mathcal{D}_*:=\overline{\mathbf{\Delta}^{\mathbf{m}}_{\mathbf{f},\mathbf{M}_\mathbf{T}}(I)(\mathcal{M}_T)},$$

and $\Delta_{\Theta_{\mathbf{f},\mathbf{T}}}:=\left(I-\Theta^*_{\mathbf{f},\mathbf{T}}\Theta_{\mathbf{f},\mathbf{T}}\right)^{1/2}$, where $\Theta_{\mathbf{f},\mathbf{T}}$ is the characteristic function of $\mathbf{T}$. Then $\mathbf{T}$ is unitarily equivalent to $\mathbb{T}:=(\mathbb{T}_1,\ldots,\mathbb{T}_k)\in\mathcal{C}^{\mathbf{m}}_{\mathbf{f}}(\mathbb{H}_{\mathbf{f},\mathbf{T}})$, where $\mathbb{T}_i:=(\mathbb{T}_{i,1},\ldots,\mathbb{T}_{i,n_i})$ and $\mathbb{T}_{i,j}$ is a bounded operator acting on the Hilbert space

$$\begin{aligned}\mathbb{H}_{\mathbf{f},\mathbf{T}}:=&\left[\left((\otimes_{i=1}^k F^2(H_{n_i}))\otimes\mathcal{D}\right)\oplus\overline{\Delta_{\Theta_{\mathbf{f},\mathbf{T}}}((\otimes_{i=1}^k F^2(H_{n_i}))\otimes\mathcal{D}_*)}\right]\\&\ominus\left\{\Theta_{\mathbf{f},\mathbf{T}}\varphi\oplus\Delta_{\Theta_{\mathbf{f},\mathbf{T}}}\varphi:\ \varphi\in(\otimes_{i=1}^k F^2(H_{n_i}))\otimes\mathcal{D}_*\right\}\end{aligned}$$

and is uniquely defined by the relation

$$\left(P_{(\otimes_{i=1}^k F^2(H_{n_i}))\otimes\mathcal{D}}|_{\mathbb{H}_{\mathbf{f},\mathbf{T}}}\right)\mathbb{T}^*_{i,j}x=(\mathbf{W}^*_{i,j}\otimes I_\mathcal{D})\left(P_{(\otimes_{i=1}^k F^2(H_{n_i}))\otimes\mathcal{D}}|_{\mathbb{H}_{\mathbf{f},\mathbf{T}}}\right)x$$

for any $x\in\mathbb{H}_{\mathbf{f},\mathbf{T}}$. Here, $P_{(\otimes_{i=1}^k F^2(H_{n_i}))\otimes\mathcal{D}}$ is the orthogonal projection of the Hilbert space

$$\mathcal{K}_{\mathbf{f},\mathbf{T}}:=\left((\otimes_{i=1}^k F^2(H_{n_i}))\otimes\mathcal{D}\right)\oplus\overline{\Delta_{\Theta_{\mathbf{f},\mathbf{T}}}((\otimes_{i=1}^k F^2(H_{n_i}))\otimes\mathcal{D}_*)}$$

onto the subspace $(\otimes_{i=1}^k F^2(H_{n_i}))\otimes\mathcal{D}$.

Proof. First, we show that there is a unique unitary operator $\Gamma:\mathcal{H}\to\mathbb{H}_{\mathbf{f},\mathbf{T}}$ such that

$$\Gamma(\mathbf{K}^*_{\mathbf{f},\mathbf{T}}g)=P_{\mathbb{H}_{\mathbf{f},\mathbf{T}}}(g\oplus 0),\qquad g\in(\otimes_{i=1}^k F^2(H_{n_i}))\otimes\mathcal{D},\tag{6.1}$$

where $P_{\mathbb{H}_{\mathbf{f},\mathbf{T}}}$ the orthogonal projection of $\mathcal{K}_{\mathbf{f},\mathbf{T}}$ onto the subspace $\mathbb{H}_{\mathbf{f},\mathbf{T}}$. Indeed, note that the operator $\Phi:(\otimes_{i=1}^k F^2(H_{n_i}))\otimes\mathcal{D}\to\mathcal{K}_{\mathbf{f},\mathbf{T}}$ defined by

$$\Phi\varphi:=\Theta_{\mathbf{f},\mathbf{T}}\varphi\oplus\Delta_{\Theta_{\mathbf{f},\mathbf{T}}}\varphi,\quad \varphi\in(\otimes_{i=1}^k F^2(H_{n_i}))\otimes\mathcal{D}_*,$$

is an isometry and

$$\Phi^*(g\oplus 0)=\Theta^*_{\mathbf{f},\mathbf{T}}g,\qquad g\in(\otimes_{i=1}^k F^2(H_{n_i}))\otimes\mathcal{D}.\tag{6.2}$$

This leads to

$$\|g\|^2 = \|P_{\mathbb{H}_{\mathbf{f},\mathbf{T}}}(g \oplus 0)\|^2 + \|\Phi\Phi^*(g \oplus 0)\|^2 = \|P_{\mathbb{H}_{\mathbf{f},\mathbf{T}}}(g \oplus 0)\|^2 + \|\Theta_{\mathbf{f},\mathbf{T}}^* g\|^2$$

for any $g \in (\otimes_{i=1}^k F^2(H_{n_i})) \otimes \mathcal{D}$. Now, taking into account that

$$\|\mathbf{K}_{\mathbf{f},\mathbf{T}}^* g\|^2 + \|\Theta_{\mathbf{f},\mathbf{T}}^* g\|^2 = \|g\|^2, \quad g \in (\otimes_{i=1}^k F^2(H_{n_i})) \otimes \mathcal{D},$$

we deduce that

$$\|\mathbf{K}_{\mathbf{f},\mathbf{T}}^* g\| = \|P_{\mathbb{H}_{\mathbf{f},\mathbf{T}}}(g \oplus 0)\|, \quad g \in (\otimes_{i=1}^k F^2(H_{n_i})) \otimes \mathcal{D}. \tag{6.3}$$

Since the k-tuple $\mathbf{T} = (T_1, \ldots, T_k)$ is completely non-coisometric, the noncommutative Berezin kernel $\mathbf{K}_{\mathbf{f},\mathbf{T}}$ is a one-to-one operator and, consequently, range $\mathbf{K}_{\mathbf{f},\mathbf{T}}^*$ is dense in $\mathcal{H}$. Now, let $x \in \mathbb{H}_{\mathbf{f},\mathbf{T}}$ and assume that $\left\langle x, P_{\mathbb{H}_{\mathbf{f},\mathbf{T}}}(g \oplus 0)\right\rangle = 0$ for any $g \in (\otimes_{i=1}^k F^2(H_{n_i})) \otimes \mathcal{D}$. Using the definition of $\mathbb{H}_{\mathbf{f},\mathbf{T}}$ and the fact that $\mathcal{K}_{\mathbf{f},\mathbf{T}}$ coincides with the span of all vectors $g \oplus 0$ for $g \in (\otimes_{i=1}^k F^2(H_{n_i})) \otimes \mathcal{D}$ and $\Theta_{\mathbf{f},\mathbf{T}}\varphi \oplus \Delta_{\Theta_{\mathbf{f},\mathbf{T}}}\varphi$ for $\varphi \in (\otimes_{i=1}^k F^2(H_{n_i})) \otimes \mathcal{D}$, we deduce that $x = 0$. This shows that

$$\mathbb{H}_{\mathbf{f},\mathbf{T}} = \left\{P_{\mathbb{H}_{\mathbf{f},\mathbf{T}}}(g \oplus 0) : \ g \in (\otimes_{i=1}^k F^2(H_{n_i})) \otimes \mathcal{D}\right\}$$

Using relation (6.3), we conclude that there is a unique unitary operator Γ satisfying relation (6.1). For each $i \in \{1, \ldots, k\}$ and $j \in \{1, \ldots, n_i\}$, let $\mathbb{T}_{i,j} : \mathbb{H}_{\mathbf{f},\mathbf{T}} \to \mathbb{H}_{\mathbf{f},\mathbf{T}}$ be defined by

$$\mathbb{T}_{i,j} := \Gamma T_{i,j} \Gamma^*, \qquad i \in \{1, \ldots, k\}, j \in \{1, \ldots, n_i\}.$$

In what follows, we prove that

$$\left(P_{(\otimes_{i=1}^k F^2(H_{n_i}))\otimes\mathcal{D}}|_{\mathbb{H}_{\mathbf{f},\mathbf{T}}}\right) \mathbb{T}_{i,j}^* x = (\mathbf{W}_{i,j}^* \otimes I_{\mathcal{D}}) \left(P_{(\otimes_{i=1}^k F^2(H_{n_i}))\otimes\mathcal{D}}|_{\mathbb{H}_{\mathbf{f},\mathbf{T}}}\right) x \tag{6.4}$$

for any $i \in \{1, \ldots, k\}$, $j \in \{1, \ldots, n_i\}$, and $x \in \mathbb{H}_{\mathbf{f},\mathbf{T}}$. Using relations (6.1) and (6.2), and the fact that Φ is an isometry, we deduce that

$$\begin{aligned} P_{(\otimes_{i=1}^k F^2(H_{n_i}))\otimes\mathcal{D}} \Gamma \mathbf{K}_{\mathbf{f},\mathbf{T}}^* g &= P_{(\otimes_{i=1}^k F^2(H_{n_i}))\otimes\mathcal{D}} P_{\mathbb{H}_{\mathbf{f},\mathbf{T}}}(g \oplus 0) = g - P_{(\otimes_{i=1}^k F^2(H_{n_i}))\otimes\mathcal{D}} \Phi\Phi^*(g \oplus 0) \\ &= g - \Theta_{\mathbf{f},\mathbf{T}}\Theta_{\mathbf{f},\mathbf{T}}^* g = \mathbf{K}_{\mathbf{f},\mathbf{T}}\mathbf{K}_{\mathbf{f},\mathbf{T}}^* g \end{aligned}$$

for any $g \in (\otimes_{i=1}^k F^2(H_{n_i})) \otimes \mathcal{D}$. Taking into account that the range of $\mathbf{K}_{\mathbf{f},\mathbf{T}}^*$ is dense in $\mathcal{H}$, we deduce that

$$P_{(\otimes_{i=1}^k F^2(H_{n_i}))\otimes\mathcal{D}} \Gamma = \mathbf{K}_{\mathbf{f},\mathbf{T}}. \tag{6.5}$$

Hence, and using the fact that the noncommutative Berezin kernel $\mathbf{K}_{\mathbf{f},\mathbf{T}}$ is one-to-one, we can see that

$$P_{(\otimes_{i=1}^k F^2(H_{n_i}))\otimes\mathcal{D}}|_{\mathbb{H}_{\mathbf{f},\mathbf{T}}} = \mathbf{K}_{\mathbf{f},\mathbf{T}}\Gamma^*$$

is a one-to-one operator acting from $\mathbb{H}_{\mathbf{f},\mathbf{T}}$ to $(\otimes_{i=1}^k F^2(H_{n_i})) \otimes \mathcal{D}$. Relation (6.5) and Theorem 2.2 imply

$$\begin{aligned} \left(P_{(\otimes_{i=1}^k F^2(H_{n_i}))\otimes\mathcal{D}}|_{\mathbb{H},\mathbf{T}}\right) \mathbb{T}_{i,j}^* \Gamma h &= \left(P_{(\otimes_{i=1}^k F^2(H_{n_i}))\otimes\mathcal{D}}|_{\mathbb{H}_{\mathbf{f},\mathbf{T}}}\right) \Gamma T_{i,j}^* h = \mathbf{K}_{\mathbf{f},\mathbf{T}} T_{i,j}^* h \\ &= \left(\mathbf{W}_{i,j}^* \otimes I_{\mathcal{D}}\right) \mathbf{K}_{\mathbf{f},\mathbf{T}} h = \left(\mathbf{W}_{i,j}^* \otimes I_{\mathcal{D}}\right) \left(P_{(\otimes_{i=1}^k F^2(H_{n_i}))\otimes\mathcal{D}}|_{\mathbb{H}_{\mathbf{f},\mathbf{T}}}\right) \Gamma h \end{aligned}$$

for any $i \in \{1, \ldots, k\}$, $j \in \{1, \ldots, n_i\}$, and $h \in \mathcal{H}$. Now, we can deduce relation (6.4). Note that, since the operator $P_{(\otimes_{i=1}^k F^2(H_{n_i}))\otimes\mathcal{D}}|_{\mathbb{H}_{\mathbf{f},\mathbf{T}}}$ is one-to-one, the relation (6.4) uniquely determines each operator $\mathbb{T}_{i,j}^*$ for all $i \in \{1, \ldots, k\}$ and $j \in \{1, \ldots, n_i\}$. This completes the proof. □

In what follows, we show that the characteristic function $\Theta_{\mathbf{f},\mathbf{T}}$ is a complete unitary invariant for the completely non-coisometric part of the noncommutative domain $\mathcal{C}_{\mathbf{f}}^{\mathbf{m}}$.

Theorem 6.5. *Let $\mathbf{T} := (T_1, \ldots, T_k) \in \mathcal{C}_{\mathbf{f}}^{\mathbf{m}}(\mathcal{H})$ and $\mathbf{T}' := (T_1', \ldots, T_k') \in \mathcal{C}_{\mathbf{f}}^{\mathbf{m}}(\mathcal{H}')$ be two completely non-coisometric k-tuples. Then $\mathbf{T}$ and $\mathbf{T}'$ are unitarily equivalent if and only if their characteristic functions $\Theta_{\mathbf{f},\mathbf{T}}$ and $\Theta_{\mathbf{f},\mathbf{T}'}$ coincide.*

Proof. Assume that the k-tuples $\mathbf{T}$ and $\mathbf{T}'$ are unitarily equivalent and let $U:\mathcal{H}\to\mathcal{H}'$ be a unitary operator such that $T_{i,j}=U^*T'_{i,j}U$ for any $i\in\{1,\dots,k\}$ and $j\in\{1,\dots,n_i\}$. It is easy to see that $U\mathbf{\Delta}^{\mathbf{m}}_{\mathbf{f},\mathbf{T}}(I)=\mathbf{\Delta}^{\mathbf{m}}_{\mathbf{f},\mathbf{T}'}(I)U$ and, consequently, $U\mathcal{D}=\mathcal{D}'$, where

$$\mathcal{D}:=\overline{\mathbf{\Delta}^{\mathbf{m}}_{\mathbf{f},\mathbf{T}}(I)(\mathcal{H})},\qquad \mathcal{D}':=\overline{\mathbf{\Delta}^{\mathbf{m}}_{\mathbf{f},\mathbf{T}'}(I)(\mathcal{H}')}.$$

Using the definition of the noncommutative Berezin kernel associated with $\mathbf{D}^{\mathbf{m}}_{\mathbf{f}}$, one can easily check that $(I_{\otimes_{i=1}^k F^2(H_{n_i})}\otimes U)\mathbf{K}_{\mathbf{f},\mathbf{T}}=\mathbf{K}_{\mathbf{f},\mathbf{T}'}U$. This implies

$$(I_{\otimes_{i=1}^k F^2(H_{n_i})}\otimes U)(I-\mathbf{K}_{\mathbf{f},\mathbf{T}}\mathbf{K}^*_{\mathbf{f},\mathbf{T}})(I_{\otimes_{i=1}^k F^2(H_{n_i})}\otimes U)=I-\mathbf{K}_{\mathbf{f},\mathbf{T}'}\mathbf{K}^*_{\mathbf{f},\mathbf{T}'}\tag{6.6}$$

and $(I_{\otimes_{i=1}^k F^2(H_{n_i})}\otimes U)\mathcal{M}_{\mathbf{T}}=\mathcal{M}_{\mathbf{T}'}$, where $\mathcal{M}_{\mathbf{T}}:=\overline{\operatorname{range}(I-\mathbf{K}_{\mathbf{f},\mathbf{T}}\mathbf{K}^*_{\mathbf{f},\mathbf{T}})}$ and $\mathcal{M}_{T'}$ is defined similarly. Recall that $\mathbf{M}_{\mathbf{T}}:=(M_1,\dots,M_k)$ is the k-tuple with $M_i:=(M_{i,1},\dots,M_{i,n_i})$ and $M_{i,j}\in B(\mathcal{M}_{\mathbf{T}})$, and it is given by $M_{i,j}:=A^*_{i,j}$, where $A_{i,j}\in B(\mathcal{M}_{\mathbf{T}})$ is uniquely defined by

$$A_{i,j}\left[(I-\mathbf{K}_{\mathbf{f},\mathbf{T}}\mathbf{K}^*_{\mathbf{f},\mathbf{T}})^{1/2}x\right]:=(I-\mathbf{K}_{\mathbf{f},\mathbf{T}}\mathbf{K}^*_{\mathbf{f},\mathbf{T}})^{1/2}(\mathbf{W}_{i,j}\otimes I)x$$

for any $x\in(\otimes_{i=1}^k F^2(H_{n_i}))\otimes\overline{\mathbf{\Delta}^{\mathbf{m}}_{\mathbf{f},\mathbf{T}}(I)(\mathcal{H})}$. Similarly, we define the k-tuple $\mathbf{M}_{\mathbf{T}'}$ and the operators $A'_{i,j}\in B(\mathcal{M}_{\mathbf{T}'})$. Note that

$$\begin{aligned}A_{i,j}(I-\mathbf{K}_{\mathbf{f},\mathbf{T}}\mathbf{K}^*_{\mathbf{f},\mathbf{T}})^{1/2}x&=(I_{\otimes_{i=1}^k F^2(H_{n_i})}\otimes U^*)A'_{i,j}(I-\mathbf{K}_{\mathbf{f},\mathbf{T}'}\mathbf{K}^*_{\mathbf{f},\mathbf{T}'})^{1/2}(I_{\otimes_{i=1}^k F^2(H_{n_i})}\otimes U^*)x\\&=(I_{\otimes_{i=1}^k F^2(H_{n_i})}\otimes U^*)A'_{i,j}(I_{\otimes_{i=1}^k F^2(H_{n_i})}\otimes U)(I-\mathbf{K}_{\mathbf{f},\mathbf{T}}\mathbf{K}^*_{\mathbf{f},\mathbf{T}})^{1/2}x\end{aligned}$$

for any $x\in(\otimes_{i=1}^k F^2(H_{n_i}))\otimes\overline{\mathbf{\Delta}^{\mathbf{m}}_{\mathbf{f},\mathbf{T}}(I)(\mathcal{H})}$. Hence, we deduce that

$$A_{i,j}=(I_{\otimes_{i=1}^k F^2(H_{n_i})}\otimes U^*)A'_{i,j}(I_{\otimes_{i=1}^k F^2(H_{n_i})}\otimes U).$$

Now, we can see that $(I_{\otimes_{i=1}^k F^2(H_{n_i})}\otimes U)\mathcal{D}_*=\mathcal{D}'_*$, where $\mathcal{D}_*:=\overline{\mathbf{\Delta}^{\mathbf{m}}_{\mathbf{f},\mathbf{M}_{\mathbf{T}}}(I)(\mathcal{M}_T)}$ and $\mathcal{D}'_*$ is defined similarly. We introduce the unitary operators τ and τ' by setting

$$\tau:=U|_{\mathcal{D}}:\mathcal{D}\to\mathcal{D}'\quad\text{and}\quad \tau_*:=(I_{\otimes_{i=1}^k F^2(H_{n_i})}\otimes U)|_{\mathcal{D}_*}:\mathcal{D}_*\to\mathcal{D}'_*.$$

Using the definition of the characteristic function, it is easy to show that

$$(I_{\otimes_{i=1}^k F^2(H_{n_i})}\otimes\tau)\Theta_{\mathbf{f},\mathbf{T}}=\Theta_{\mathbf{f},\mathbf{T}'}(I_{\otimes_{i=1}^k F^2(H_{n_i})}\otimes\tau_*).$$

To prove the converse, assume that the characteristic functions of $\mathbf{T}$ and $\mathbf{T}'$ coincide. Then there exist unitary operators $\tau:\mathcal{D}\to\mathcal{D}'$ and $\tau_*:\mathcal{D}_*\to\mathcal{D}'_*$ such that

$$(I_{\otimes_{i=1}^k F^2(H_{n_i})}\otimes\tau)\Theta_{\mathbf{f},\mathbf{T}}=\Theta_{\mathbf{f},\mathbf{T}'}(I_{\otimes_{i=1}^k F^2(H_{n_i})}\otimes\tau_*).$$

It is clear that this relation implies

$$\Delta_{\Theta_{\mathbf{f},\mathbf{T}}}=\left(I_{\otimes_{i=1}^k F^2(H_{n_i})}\otimes\tau_*\right)^*\Delta_{\Theta_{\mathbf{f},\mathbf{T}'}}\left(I_{(\otimes_{i=1}^k F^2(H_{n_i})}\otimes\tau_*\right)$$

and

$$\left(I_{\otimes_{i=1}^k F^2(H_{n_i})}\otimes\tau_*\right)\overline{\Delta_{\Theta_{\mathbf{f},\mathbf{T}}}((\otimes_{i=1}^k F^2(H_{n_i})\otimes\mathcal{D}_*)}=\overline{\Delta_{\Theta_{\mathbf{f},\mathbf{T}'}}(\otimes_{i=1}^k F^2(H_{n_i})\otimes\mathcal{D}'_*)}.$$

Define now the unitary operator $U:\mathcal{K}_{\mathbf{f},\mathbf{T}}\to\mathcal{K}_{\mathbf{f},\mathbf{T}'}$ by setting

$$U:=(I_{\otimes_{i=1}^k F^2(H_{n_i})}\otimes\tau)\oplus(I_{\otimes_{i=1}^k F^2(H_{n_i})}\otimes\tau_*).$$

Note that the operator $\Phi:(\otimes_{i=1}^k F^2(H_{n_i})\otimes\mathcal{D}_*\to\mathcal{K}_{\mathbf{f},\mathbf{T}}$, defined by

$$\Phi\varphi:=\Theta_{\mathbf{f},\mathbf{T}}\varphi\oplus\Delta_{\Theta_{\mathbf{f},\mathbf{T}}}\varphi,\quad \varphi\in(\otimes_{i=1}^k F^2(H_{n_i})\otimes\mathcal{D}_*,$$

and the corresponding Φ' satisfy the following relations:

$$U\Phi\left(I_{\otimes_{i=1}^k F^2(H_{n_i})}\otimes\tau_*\right)^*=\Phi'\tag{6.7}$$

and

$$\left(I_{\otimes_{i=1}^k F^2(H_{n_i})}\otimes\tau\right)P^{\mathcal{K}_{\mathbf{f},\mathbf{T}}}_{\otimes_{i=1}^k F^2(H_{n_i})\otimes\mathcal{D}}U^*=P^{\mathcal{K}_{\mathbf{f},\mathbf{T}'}}_{\otimes_{i=1}^k F^2(H_{n_i})\otimes\mathcal{D}'},\tag{6.8}$$

where $P^{\mathcal{K}_{\mathbf{f},\mathbf{T}}}_{(\otimes_{i=1}^k F^2(H_{n_i})\otimes\mathcal{D}}$ is the orthogonal projection of $\mathcal{K}_{\mathbf{f},\mathbf{T}}$ onto $(\otimes_{i=1}^k F^2(H_{n_i})\otimes\mathcal{D}$. Note also that relation (6.7) implies

$$\begin{aligned}U\mathbb{H}_{\mathbf{f},\mathbf{T}} &= U\mathcal{K}_{\mathbf{f},\mathbf{T}} \ominus U\Phi((\otimes_{i=1}^k F^2(H_{n_i})\otimes\mathcal{D}_*)\\ &= \mathcal{K}_{\mathbf{f},\mathbf{T}'} \ominus \Phi'(I_{\otimes_{i=1}^k F^2(H_{n_i})}\otimes\tau_*)((\otimes_{i=1}^k F^2(H_{n_i})\otimes\mathcal{D}_*)\\ &= \mathcal{K}_{\mathbf{f},\mathbf{T}'} \ominus \Phi'((\otimes_{i=1}^k F^2(H_{n_i})\otimes\mathcal{D}'_*).\end{aligned}$$

This shows that the operator $U|_{\mathbb{H}_{\mathbf{f},\mathbf{T}}}:\mathbb{H}_{\mathbf{f},\mathbf{T}}\to\mathbb{H}_{\mathbf{f},\mathbf{T}'}$ is unitary. Note also that

$$(\mathbf{W}_{i,j}^*\otimes I_{\mathcal{D}'})(I_{(\otimes_{i=1}^k F^2(H_{n_i})}\otimes\tau) = (I_{(\otimes_{i=1}^k F^2(H_{n_i})}\otimes\tau)(\mathbf{W}_{i,j}^*\otimes I_{\mathcal{D}}). \tag{6.9}$$

Let $\mathbb{T}:=(\mathbb{T}_1,\ldots\mathbb{T}_n)$ and $\mathbb{T}':=(\mathbb{T}'_1,\ldots\mathbb{T}'_n)$ be the model operators provided by Theorem 6.4 for $\mathbf{T}$ and $\mathbf{T}'$, respectively. Using the relation (6.4) for $\mathbf{T}'$ and $\mathbf{T}$, as well as (6.8) and (6.9), we have

$$\begin{aligned}P^{\mathbb{K}_{\mathbf{f},\mathbf{T}'}}_{(\otimes_{i=1}^k F^2(H_{n_i})\otimes\mathcal{D}'}{\mathbb{T}'_{i,j}}^*Ux &= (\mathbf{W}_{i,j}^*\otimes I_{\mathcal{D}'})P^{\mathcal{K}_{\mathbf{f},\mathbf{T}}}_{(\otimes_{i=1}^k F^2(H_{n_i})\otimes\mathcal{D}}Ux\\ &= (\mathbf{W}_{i,j}^*\otimes I_{\mathcal{D}'})(I_{\otimes_{i=1}^k F^2(H_{n_i})}\otimes\tau)P^{\mathcal{K}_{\mathbf{f},\mathbf{T}}}_{(\otimes_{i=1}^k F^2(H_{n_i})\otimes\mathcal{D}}x\\ &= (I_{\otimes_{i=1}^k F^2(H_{n_i})}\otimes\tau)(\mathbf{W}_{i,j}^*\otimes I_{\mathcal{D}})P^{\mathcal{K}_{\mathbf{f},\mathbf{T}}}_{(\otimes_{i=1}^k F^2(H_{n_i})\otimes\mathcal{D}}x\\ &= (I_{\otimes_{i=1}^k F^2(H_{n_i})}\otimes\tau)P^{\mathcal{K}_{\mathbf{f},\mathbf{T}}}_{(\otimes_{i=1}^k F^2(H_{n_i})\otimes\mathcal{D}}\mathbb{T}_i^*x\\ &= P^{\mathcal{K}_{\mathbf{f},\mathbf{T}'}}_{(\otimes_{i=1}^k F^2(H_{n_i})\otimes\mathcal{D}'}U\mathbb{T}_{i,j}^*x\end{aligned}$$

for any $i=\{1,\ldots,k\}$, $j\in\{1,\ldots,n_i\}$, and $x\in\mathbb{H}_{\mathbf{f},\mathbf{T}}$. Since $P^{\mathcal{K}_{\mathbf{f},\mathbf{T}'}}_{(\otimes_{i=1}^k F^2(H_{n_i})\otimes\mathcal{D}'}|_{\mathbb{H}_{\mathbf{f},\mathbf{T}'}}$ is an one-to-one operator (see Theorem 6.4), we obtain $\left(U|_{\mathbb{H}_{\mathbf{f},\mathbf{T}}}\right)\mathbb{T}_{i,j}^* = (\mathbb{T}'_{i,j})^*\left(U|_{\mathbb{H}_{\mathbf{f},\mathbf{T}}}\right)$. Due to Theorem 6.4, we conclude that the k-tuples $\mathbf{T}$ and $\mathbf{T}'$ are unitarily equivalent. The proof is complete. ☐

Proposition 6.6. *If $\mathbf{T}=(T_1,\ldots,T_k)\in\mathbf{D_f^m}(\mathcal{H})$, then the following statements hold.*

(i) *$\mathbf{T}$ is unitarily equivalent to $(\mathbf{W}_1\otimes I_{\mathcal{K}},\ldots,\mathbf{W}_k\otimes I_{\mathcal{K}})$ for some Hilbert space $\mathcal{K}$ if and only if $\mathbf{T}\in\mathcal{C}_\mathbf{f}^\mathbf{m}(\mathcal{H})$ is completely non-coisometric and the characteristic function $\Theta_{\mathbf{f},\mathbf{T}}=0$.*

(ii) *If $\mathbf{T}\in\mathcal{C}_\mathbf{f}^\mathbf{m}(\mathcal{H})$, then $\Theta_{\mathbf{f},\mathbf{T}}$ has dense range if and only if there is no nonzero vector $h\in\mathcal{H}$ such that*

$$\lim_{q=(q_1,\ldots,q_k)\in\mathbb{N}^k}\left\langle (id-\Phi_{f_1,T_1}^{q_1})\cdots(id-\Phi_{f_k,T_k}^{q_k})(I_{\mathcal{H}})h,h\right\rangle = \|h\|.$$

Proof. Note that if $\mathbf{T}=(\mathbf{W}_1\otimes I_{\mathcal{K}},\ldots,\mathbf{W}_k\otimes I_{\mathcal{K}})$ for some Hilbert space $\mathcal{K}$, then $\mathbf{K}_{\mathbf{f},\mathbf{T}}=I$. Since $\mathbf{K}_{\mathbf{f},\mathbf{T}}\mathbf{K}_{\mathbf{f},\mathbf{T}}^*+\Theta_{\mathbf{f},\mathbf{T}}\Theta_{\mathbf{f},\mathbf{T}}^*=I$, we deduce that $\Theta_{\mathbf{f},\mathbf{T}}=0$. Conversely, if $\mathbf{T}\in\mathcal{C}_\mathbf{f}^\mathbf{m}(\mathcal{H})$ is completely non-coisometric and the characteristic function $\Theta_{\mathbf{f},\mathbf{T}}=0$, then $\mathbf{K}_{\mathbf{f},\mathbf{T}}\mathbf{K}_{\mathbf{f},\mathbf{T}}^*=I$. Using Theorem 6.4, the result follows.

Due to Theorem 2.2, the condition in item (ii) is equivalent to $\ker(I-\mathbf{K}_{\mathbf{f},\mathbf{T}}^*\mathbf{K}_{\mathbf{f},\mathbf{T}})=\{0\}$, which is equivalent to $\ker(I-\mathbf{K}_{\mathbf{f},\mathbf{T}}\mathbf{K}_{\mathbf{f},\mathbf{T}}^*)=\{0\}$ and, therefore, to $\ker\Theta_{\mathbf{f},\mathbf{T}}\Theta_{\mathbf{f},\mathbf{T}}^*=\{0\}$. Hence, the result follows. The proof is complete. ☐

7. Dilation theory on noncommutative polydomains

We develop a dilation theory on the noncommutative polydomain $\mathbf{D_q^m}(\mathcal{H})$ and obtain Wold type decompositions for non-degenerate $*$-representations of the C^*-algebra $C^*(\mathbf{W}_{i,j})$.

We recall that $\mathcal{P}(\mathbf{W})$ is the set of all polynomials $p(\mathbf{W}_{i,j})$ in the operators $\mathbf{W}_{i,j}$, $i\in\{1,\ldots,k\}$, $j\in\{1,\ldots,n_i\}$, and the identity.

Lemma 7.1. *Let $\mathbf{q}=(q_1,\ldots,q_k)$ be a k-tuple of positive regular noncommutative polynomials and let $\mathbf{W}=(\mathbf{W}_1,\ldots,\mathbf{W}_k)$ be the universal model associated with the noncommutative polydomain $\mathbf{D_q^m}$. Then all the compact operators in $B(\otimes_{i=1}^k F^2(H_{n_i}))$ are contained in the operator space*

$$\mathcal{S}:=\overline{\operatorname{span}}\{p(\mathbf{W}_{i,j})q(\mathbf{W}_{i,j})^*:\ p(\mathbf{W}_{i,j}),q(\mathbf{W}_{i,j})\in\mathcal{P}(\mathbf{W})\},$$

where the closure is in the operator norm.

Proof. According to Lemma 2.1, we have

$$(I-\Phi_{q_1,\mathbf{W}_1})^{m_1}\cdots(I-\Phi_{q_k,\mathbf{W}_k})^{m_k}(I)=\mathbf{P}_{\mathbb{C}}, \tag{7.1}$$

where $\mathbf{P}_{\mathbb{C}}$ is the orthogonal projection from $\otimes_{i=1}^k F^2(H_{n_i})$ onto $\mathbb{C}1\subset\otimes_{i=1}^k F^2(H_{n_i})$. Fix

$$g(\mathbf{W}_{i,j}):=\sum_{\substack{\beta_1\in\mathbb{F}_{n_1}^+,\ldots,\beta_k\in\mathbb{F}_{n_k}^+\\ |\beta_1|+\cdots+|\beta_k|\le n}} d_{\beta_1,\ldots,\beta_k}\mathbf{W}_{1,\beta_1}\cdots\mathbf{W}_{k,\beta_k}\quad\text{and}\quad \xi:=\sum_{\beta_1\in\mathbb{F}_{n_1}^+,\ldots,\beta_k\in\mathbb{F}_{n_k}^+} c_{\beta_1,\ldots,\beta_k}e^1_{\beta_1}\otimes\cdots\otimes e^k_{\beta_k}$$

and note that $\mathbf{P}_{\mathbb{C}}g(\mathbf{W}_{i,j})^*\xi=\left<\xi,g(\mathbf{W}_{i,j})(1)\right>$. Consequently, we have

$$\chi(\mathbf{W}_{i,j})\mathbf{P}_{\mathbb{C}}g(\mathbf{W}_{i,j})^*\xi=\left<\xi,g(\mathbf{W}_{i,j})(1)\right>\chi(\mathbf{W}_{i,j})(1) \tag{7.2}$$

for any polynomial $\chi(\mathbf{W}_{i,j})$. Using relation (7.1), we deduce that the operator $\chi(\mathbf{W}_{i,j})\mathbf{P}_{\mathbb{C}}g(\mathbf{W}_{i,j})^*$ has rank one and it is in the operator space $\mathcal{S}$. On the other hand, due to the fact that the set of all vectors of the form $\sum_{\substack{\beta_1\in\mathbb{F}_{n_1}^+,\ldots,\beta_k\in\mathbb{F}_{n_k}^+\\ |\beta_1|+\cdots+|\beta_k|\le n}} d_{\beta_1,\ldots,\beta_k}\mathbf{W}_{1,\beta_1}\cdots\mathbf{W}_{k,\beta_k}(1)$ with $n\in\mathbb{N}$, $d_{\beta_1,\ldots,\beta_k}\in\mathbb{C}$, is dense in $\otimes_{i=1}^k F^2(H_{n_i})$, relation (7.2) implies that all the compact operators in $B(\otimes_{i=1}^k F^2(H_{n_i}))$ are contained in $\mathcal{S}$. This completes the proof. $\square$

Let $C^*(\Gamma)$ be the C^*-algebra generated by a set of operators $\Gamma\subset B(\mathcal{K})$ and the identity. A subspace $\mathcal{H}\subset\mathcal{K}$ is called $*$-cyclic for Γ if $\mathcal{K}=\overline{\text{span}}\{Xh, X\in C^*(\Gamma), h\in\mathcal{H}\}$. The main result of this section is the following dilation theorem for the elements of the noncommutative polydomain $\mathbf{D}_{\mathbf{q}}^{\mathbf{m}}(\mathcal{H})$.

Theorem 7.2. *Let $\mathbf{q}=(q_1,\ldots,q_k)$ be a k-tuple of positive regular noncommutative polynomials and let $\mathbf{W}=(\mathbf{W}_1,\ldots,\mathbf{W}_k)$ be the universal model associated with the abstract noncommutative polydomain $\mathbf{D}_{\mathbf{q}}^{\mathbf{m}}$. If $\mathbf{T}=(T_1,\ldots,T_k)$ is a k-tuple in $\mathbf{D}_{\mathbf{q}}^{\mathbf{m}}(\mathcal{H})$, then there exists a $*$-representation $\pi:C^*(\mathbf{W}_{i,j})\to B(\mathcal{K}_\pi)$ on a separable Hilbert space $\mathcal{K}_\pi$, which annihilates the compact operators and*

$$(I-\Phi_{q_1,\pi(\mathbf{W}_1)})\cdots(I-\Phi_{q_k,\pi(\mathbf{W}_k)})(I_{\mathcal{K}_\pi})=0,$$

where $\pi(\mathbf{W}_i):=(\pi(\mathbf{W}_{i,1}),\ldots,\pi(\mathbf{W}_{i,n_i}))$, such that $\mathcal{H}$ can be identified with a $*$-cyclic co-invariant subspace of

$$\tilde{\mathcal{K}}:=\left[(\otimes_{i=1}^k F^2(H_{n_i}))\otimes\overline{\mathbf{\Delta}_{\mathbf{q},\mathbf{T}}^{\mathbf{m}}(I)(\mathcal{H})}\right]\oplus\mathcal{K}_\pi$$

under each operator

$$V_{i,j}:=\begin{bmatrix}\mathbf{W}_{i,j}\otimes I_{\overline{\mathbf{\Delta}_{\mathbf{q},\mathbf{T}}^{\mathbf{m}}(I)(\mathcal{H})}} & 0\\ 0 & \pi(\mathbf{W}_{i,j})\end{bmatrix},\qquad i\in\{1,\ldots,k\}, j\in\{1,\ldots,n_i\},$$

where $\mathbf{\Delta}_{\mathbf{q},\mathbf{T}}^{\mathbf{m}}(I):=(id-\Phi_{q_1,T_1})^{m_1}\cdots(id-\Phi_{q_k,T_k})^{m_k}(I)$, and such that $T_{i,j}^*=V_{i,j}^*|_{\mathcal{H}}$ for all $i\in\{1,\ldots,k\}$ and $j\in\{1,\ldots,n_i\}$.

Proof. Applying Arveson extension theorem [3] to the map $\Psi_{\mathbf{q},\mathbf{T}}$ of Theorem 2.4, we find a unital completely positive linear map $\Psi_{\mathbf{q},\mathbf{T}}:C^*(\mathbf{W}_{i,j})\to B(\mathcal{H})$ such that $\Psi_{\mathbf{q},\mathbf{T}}(\mathbf{W}_{(\alpha)}\mathbf{W}_{(\beta)})^*=\mathbf{T}_{(\alpha)}\mathbf{T}_{(\beta)}^*$, where $\mathbf{T}_{(\alpha)}:=T_{1,\alpha_1}\cdots T_{k,\alpha_k}$ for $(\alpha):=(\alpha_1,\ldots,\alpha_k)\in\mathbb{F}_{n_1}^+\times\cdots\times\mathbb{F}_{n_k}^+$, and $\mathbf{W}_{(\alpha)}$ is defined similarly. Let $\tilde{\pi}:C^*(\mathbf{W}_{i,j})\to B(\tilde{\mathcal{K}})$ be the minimal Stinespring dilation [53] of $\Psi_{\mathbf{q},\mathbf{T}}$. Then we have

$$\Psi_{\mathbf{q},\mathbf{T}}(X)=P_{\mathcal{H}}\tilde{\pi}(X)|\mathcal{H},\quad X\in C^*(\mathbf{W}_{i,j}),$$

and $\tilde{\mathcal{K}}=\overline{\text{span}}\{\tilde{\pi}(X)h:\ X\in C^*(\mathbf{W}_{i,j}), h\in\mathcal{H}\}$. Now, we prove that that $P_{\mathcal{H}}\tilde{\pi}(\mathbf{W}_{(\alpha)})|_{\mathcal{H}^\perp}=0$ for any $(\alpha):=(\alpha_1,\ldots,\alpha_k)\in\mathbb{F}_{n_1}^+\times\cdots\times\mathbb{F}_{n_k}^+$. Indeed, we have

$$\begin{aligned}\Psi_{\mathbf{q},\mathbf{T}}(\mathbf{W}_{(\alpha)}\mathbf{W}_{(\alpha)}^*)&=\mathbf{T}_{(\alpha)}\mathbf{T}_{(\alpha)}^*=P_{\mathcal{H}}\tilde{\pi}(\mathbf{W}_{(\alpha)})\tilde{\pi}(\mathbf{W}_{(\alpha)}^*)|_{\mathcal{H}}\\ &=P_{\mathcal{H}}\tilde{\pi}(\mathbf{W}_{(\alpha)})(P_{\mathcal{H}}+P_{\mathcal{H}^\perp})\tilde{\pi}(\mathbf{W}_{(\alpha)}^*)|_{\mathcal{H}}\\ &=\Psi_{\mathbf{q},\mathbf{T}}(\mathbf{W}_{(\alpha)}\mathbf{W}_{(\alpha)}^*)+(P_{\mathcal{H}}\tilde{\pi}(\mathbf{W}_{(\alpha)})|_{\mathcal{H}^\perp})(P_{\mathcal{H}}\tilde{\pi}(\mathbf{W}_{(\alpha)})|_{\mathcal{H}^\perp})^*.\end{aligned}$$

Consequently, we deduce that $P_{\mathcal{H}}\tilde{\pi}(\mathbf{W}_{(\alpha)})|_{\mathcal{H}^\perp}=0$ and, therefore, $\mathcal{H}$ is an invariant subspace under each operator $\tilde{\pi}(\mathbf{W}_{i,j})^*$ and

$$\tilde{\pi}(\mathbf{W}_{i,j})^*|_{\mathcal{H}}=\Psi_{\mathbf{q},\mathbf{T}}(\mathbf{W}_{i,j}^*)=T_{i,j}^* \tag{7.3}$$

for any $i \in \{1, \dots, k\}$ and $j \in \{1, \dots, n_i\}$.

According to Theorem 7.1, all the compact operators $\mathcal{C}(\otimes_{i=1}^k F^2(H_{n_i}))$ in $B(\otimes_{i=1}^k F^2(H_{n_i}))$ are contained in the C^*-algebra $C^*(\mathbf{W}_{i,j})$. Due to standard theory of representations of C^*-algebras [4], the representation $\tilde{\pi}$ decomposes into a direct sum $\tilde{\pi} = \pi_0 \oplus \pi$ on $\tilde{\mathcal{K}} = \mathcal{K}_0 \oplus \mathcal{K}_\pi$, where π_0, π are disjoint representations of $C^*(\mathbf{W}_{i,j})$ on the Hilbert spaces

$$\mathcal{K}_0 := \overline{\operatorname{span}}\{\tilde{\pi}(X)\tilde{\mathcal{K}} : \ X \in \mathcal{C}(\otimes_{i=1}^k F^2(H_{n_i}))\} \quad \text{and} \quad \mathcal{K}_\pi := \mathcal{K}_0^\perp,$$

respectively, such that π annihilates the compact operators in $B(\otimes_{i=1}^k F^2(H_{n_i}))$, and π_0 is uniquely determined by the action of $\tilde{\pi}$ on the ideal $\mathcal{C}(\otimes_{i=1}^k F^2(H_{n_i}))$ of compact operators. Since every representation of $\mathcal{C}(\otimes_{i=1}^k F^2(H_{n_i}))$ is equivalent to a multiple of the identity representation, we deduce that

$$\mathcal{K}_0 \simeq (\otimes_{i=1}^k F^2(H_{n_i})) \otimes \mathcal{G}, \quad \pi_0(X) = X \otimes I_{\mathcal{G}}, \quad X \in C^*(\mathbf{W}_{i,j}), \tag{7.4}$$

for some Hilbert space $\mathcal{G}$. Using Theorem 7.1 and its proof, one can easily see that

$$\begin{aligned}
\mathcal{K}_0 &:= \overline{\operatorname{span}}\{\tilde{\pi}(X)\tilde{\mathcal{K}} : \ X \in \mathcal{C}(\otimes_{i=1}^k F^2(H_{n_i}))\} \\
&= \overline{\operatorname{span}}\{\tilde{\pi}(\mathbf{W}_{(\alpha)}\mathbf{P}_{\mathbb{C}}\mathbf{W}_{(\beta)}^*)\tilde{\mathcal{K}} : \ (\alpha), (\beta) \in \mathbb{F}_{n_1}^+ \times \cdots \times \mathbb{F}_{n_k}^+\} \\
&= \overline{\operatorname{span}}\left\{\tilde{\pi}(\mathbf{W}_{(\alpha)})\left[(I - \Phi_{q_1,\tilde{\pi}(\mathbf{W}_1)})^{m_1} \cdots (I - \Phi_{q_k,\tilde{\pi}(\mathbf{W}_k)})^{m_k}(I_{\tilde{\mathcal{K}}})\right]\tilde{\mathcal{K}} : \ (\alpha) \in \mathbb{F}_{n_1}^+ \times \cdots \times \mathbb{F}_{n_k}^+\right\}.
\end{aligned}$$

Since $(I - \Phi_{q_1,\mathbf{W}_1})^{m_1} \cdots (I - \Phi_{q_k,\mathbf{W}_k}^{m_k})(I) = \mathbf{P}_{\mathbb{C}}$, is a projection of rank one in $C^*(\mathbf{W}_{i,j})$, we deduce that $(I - \Phi_{q_1,\pi(\mathbf{W}_1)})^{m_1} \cdots (I - \Phi_{q_k,\pi(\mathbf{W}_k)})^{m_k}(I_{\mathcal{K}_\pi}) = 0$ and $\dim \mathcal{G} = \dim [\operatorname{range} \tilde{\pi}(\mathbf{P}_{\mathbb{C}})]$. On the other hand, since the Stinespring representation $\tilde{\pi}$ is minimal, we can use the proof of Theorem 7.1 to deduce that

$$\operatorname{range} \tilde{\pi}(\mathbf{P}_{\mathbb{C}}) = \overline{\operatorname{span}}\{\tilde{\pi}(\mathbf{P}_{\mathbb{C}})\tilde{\pi}(\mathbf{W}_{(\beta)}^*)h : \ (\beta) \in \mathbb{F}_{n_1}^+ \times \cdots \times \mathbb{F}_{n_k}^+, h \in \mathcal{H}\}.$$

Indeed, we have

$$\begin{aligned}
\operatorname{range} \tilde{\pi}(\mathbf{P}_{\mathbb{C}}) &= \overline{\operatorname{span}}\{\tilde{\pi}(\mathbf{P}_{\mathbb{C}})\tilde{\pi}(X)h : \ X \in C^*(\mathbf{W}_{i,j}), h \in \mathcal{H}\} \\
&= \overline{\operatorname{span}}\{\tilde{\pi}(\mathbf{P}_{\mathbb{C}})\tilde{\pi}(Y)h : \ Y \in \mathcal{C}(\otimes_{i=1}^k F^2(H_{n_i})), h \in \mathcal{H}\} \\
&= \overline{\operatorname{span}}\{\tilde{\pi}(\mathbf{P}_{\mathbb{C}})\tilde{\pi}(\mathbf{W}_{(\alpha)}\mathbf{P}_{\mathbb{C}}\mathbf{W}_{(\beta)}^*)h : \ (\alpha), (\beta) \in \mathbb{F}_{n_1}^+ \times \cdots \times \mathbb{F}_{n_k}^+, h \in \mathcal{H}\} \\
&= \overline{\operatorname{span}}\{\tilde{\pi}(\mathbf{P}_{\mathbb{C}})\tilde{\pi}(\mathbf{W}_{(\beta)}^*)h : \ (\beta) \in \mathbb{F}_{n_1}^+ \times \cdots \times \mathbb{F}_{n_k}^+, h \in \mathcal{H}\}.
\end{aligned}$$

Now, using the fact that

$$\begin{aligned}
\Psi_{\mathbf{q},\mathbf{T}}(\mathbf{W}_{(\alpha)}X) &= P_{\mathcal{H}}(\tilde{\pi}(\mathbf{W}_{(\alpha)})\tilde{\pi}(X))|\mathcal{H} \\
&= (P_{\mathcal{H}}\tilde{\pi}(\mathbf{W}_{(\alpha)})|\mathcal{H})(P_{\mathcal{H}}\tilde{\pi}(X)|\mathcal{H}) \\
&= \Psi_{\mathbf{q},\mathbf{T}}(\mathbf{W}_{(\alpha)})\Psi_{\mathbf{q},\mathbf{T}}(X)
\end{aligned}$$

for any $X \in C^*(\mathbf{W}_{i,j})$ and $(\alpha) \in \mathbb{F}_{n_1}^+ \times \cdots \times \mathbb{F}_{n_k}^+$, it is easy to see that

$$\begin{aligned}
\left\langle \tilde{\pi}(\mathbf{P}_{\mathbb{C}})\tilde{\pi}(\mathbf{W}_{(\alpha)}^*)h, \tilde{\pi}(\mathbf{P}_{\mathbb{C}})\tilde{\pi}(\mathbf{W}_{(\beta)}^*)k \right\rangle &= \left\langle h, \mathbf{T}_{(\alpha)}\left[(id - \Phi_{q_1,T_1})^{m_1} \cdots (id - \Phi_{q_k,T_k})^{m_k}(I_{\mathcal{H}})\right]\mathbf{T}_{(\beta)}^* h \right\rangle \\
&= \left\langle \mathbf{\Delta}_{\mathbf{q},\mathbf{T}}^{\mathbf{m}}(I)\mathbf{T}_{(\alpha)}^* h, \mathbf{\Delta}_{\mathbf{q},\mathbf{T}}^{\mathbf{m}}(I)\mathbf{T}_{(\beta)}^* k \right\rangle
\end{aligned}$$

for any $h, k \in \mathcal{H}$ and $(\alpha), (\beta) \in \mathbb{F}_{n_1}^+ \times \cdots \times \mathbb{F}_{n_k}^+$. This implies the existence of a unitary operator $\Lambda : \operatorname{range} \tilde{\pi}(\mathbf{P}_{\mathbb{C}}) \to \overline{\mathbf{\Delta}_{\mathbf{q},\mathbf{T}}^{\mathbf{m}}(I)\mathcal{H}}$ defined by

$$\Lambda[\tilde{\pi}(\mathbf{P}_{\mathbb{C}})\tilde{\pi}(\mathbf{W}_{(\alpha)}^*)h] := \mathbf{\Delta}_{\mathbf{q},\mathbf{T}}^{\mathbf{m}}(I)\mathbf{T}_{(\alpha)}^* h, \quad h \in \mathcal{H}, \ \alpha \in \mathbb{F}_n^+.$$

This shows that

$$\dim[\operatorname{range} \pi(\mathbf{P}_{\mathbb{C}})] = \dim \overline{\mathbf{\Delta}_{\mathbf{q},\mathbf{T}}^{\mathbf{m}}(I)\mathcal{H}} = \dim \mathcal{G}.$$

Using relations (7.3) and (7.4), and identifying $\mathcal{G}$ with $\overline{\mathbf{\Delta}_{\mathbf{q},\mathbf{T}}^{\mathbf{m}}(I)\mathcal{H}}$, we obtain the required dilation. On the other hand, due to the fact that $(I - \Phi_{q_1,\pi(\mathbf{W}_1)})^{m_1} \cdots (I - \Phi_{q_k,\pi(\mathbf{W}_k)})^{m_k}(I_{\mathcal{K}_\pi}) = 0$, we can use Proposition 1.9 to deduce that $(I - \Phi_{q_1,\pi(\mathbf{W}_1)}) \cdots (I - \Phi_{q_k,\pi(\mathbf{W}_k)})(I_{\mathcal{K}_\pi}) = 0$. The proof is complete. $\square$

We remark that if we replace $\mathbf{q}=(q_1,\ldots,q_k)$, in Theorem 7.2, by a k-tuple $\mathbf{f}:=(f_1,\ldots,f)$ of positive regular free holomorphic functions we obtain a dilation theorem for any $\mathbf{T}=(T_1,\ldots,T_k)$ in $\mathbf{D_f^m}(\mathcal{H})$. More precisely, one can show that there is a $*$-representation $\tilde{\pi}: C^*(\mathbf{W}_{i,j})\to B(\tilde{\mathcal{K}})$ such that $\mathcal{H}$ is an invariant subspace under each operator $\tilde{\pi}(\mathbf{W}_{i,j})^*$ and $T_{i,j}^*=\tilde{\pi}(\mathbf{W}_{i,j})^*|_{\mathcal{H}}$ for any $i\in\{1,\ldots,k\}, j\in\{1,\ldots,n_i\}$.

On the other hand, note that, using the proof of Theorem 7.2 and due to the standard theory of representations of C^*-algebras, one can deduce the following Wold type decomposition for non-degenerate $*$-representations of the C^*-algebra $C^*(\mathbf{W}_{i,j})$.

Corollary 7.3. *Let $\mathbf{q}=(q_1,\ldots,q_k)$ be a k-tuple of positive regular noncommutative polynomials and let $\mathbf{W}=(\mathbf{W}_{i,j})$ be the universal model associated with the abstract noncommutative polydomain $\mathbf{D_q^m}$. If $\pi: C^*(\mathbf{W}_{i,j})\to B(\mathcal{K})$ is a nondegenerate $*$-representation of $C^*(\mathbf{W}_{i,j})$ on a separable Hilbert space $\mathcal{K}$, then π decomposes into a direct sum*

$$\pi=\pi_0\oplus\pi_1 \quad on \quad \mathcal{K}=\mathcal{K}_0\oplus\mathcal{K}_1,$$

where π_0 and π_1 are disjoint representations of $C^*(\mathbf{W}_{i,j})$ on the Hilbert spaces

$$\mathcal{K}_0:=\overline{\text{span}}\left\{\pi(\mathbf{W}_{(\alpha)})\left[(I-\Phi_{q_1,\pi(\mathbf{W}_1)})^{m_1}\cdots(I-\Phi_{q_k,\pi(\mathbf{W}_k)})^{m_k}(I_{\mathcal{K}})\right]\mathcal{K}:\ (\alpha)\in\mathbb{F}_{n_1}^+\times\cdots\times\mathbb{F}_{n_k}\right\}$$

and $\mathcal{K}_1:=\mathcal{K}_0^\perp$, respectively, where $\pi(\mathbf{W}_i):=(\pi(\mathbf{W}_{i,1}),\ldots,\pi(\mathbf{W}_{i,n_i}))$. Moreover, up to an isomorphism,

$$\mathcal{K}_0\simeq(\otimes_{i=1}^k F^2(H_{n_i}))\otimes\mathcal{G},\quad \pi_0(X)=X\otimes I_{\mathcal{G}}\quad for\ any\quad X\in C^*(\mathbf{W}_{i,j}),$$

where $\mathcal{G}$ is a Hilbert space with

$$\dim\mathcal{G}=\dim\left\{\text{range}\ \left[(I-\Phi_{q_1,\pi(\mathbf{W}_1)})^{m_1}\cdots(I-\Phi_{q_k,\pi(\mathbf{W}_k)})^{m_k}(I_{\mathcal{K}})\right]\right\},$$

and π_1 is a $*$-representation which annihilates the compact operators and

$$(I-\Phi_{q_1,\pi_1(\mathbf{W}_1)})\cdots(I-\Phi_{q_k,\pi_1(\mathbf{W}_k)})(I_{\mathcal{K}_1})=0.$$

If π' is another nondegenerate $*$-representation of $C^*(\mathbf{W}_{i,j})$ on a separable Hilbert space $\mathcal{K}'$, then π is unitarily equivalent to π' if and only if $\dim\mathcal{G}=\dim\mathcal{G}'$ and π_1 is unitarily equivalent to π_1'.

Note that in the particular case when $\mathbf{m}=(1,\ldots,1)$, $q_i:=Z_{i,1}+\cdots+Z_{i,n_i}$ for $i\in\{1,\ldots,k\}$, and $V_i=(V_{i,1},\ldots,V_{i,n_i})$ are row isometries such that $\mathbf{V}=(V_{i,j})$ are doubly commuting, Corollary 7.3 provides a Wold type decomposition for $\mathbf{V}$. We also remark that under the hypotheses and notations of Corollary 7.3, and setting $V_{i,j}:=\pi(\mathbf{W}_{i,j})$ for any $i\in\{1,\ldots,k\}$ and $j\in\{1,\ldots,n_i\}$, the following statements are equivalent:

(i) $\mathbf{V}:=(V_1,\ldots,V_k)$ is a pure element in $\mathbf{D_q^m}(\mathcal{K})$;
(ii) for each $i\in\{1,\ldots,k\}$, $\lim_{p\to\infty}\Phi_{q_i,V_i}^p(I)=0$ in the strong operator topology;
(iii) $\mathcal{K}:=\overline{\text{span}}\left\{V_{(\alpha)}\left[(I-\Phi_{q_1,V_1})^{m_1}\cdots(I-\Phi_{q_k,V_k})^{m_k}(I_{\mathcal{K}})\right](\mathcal{K}):\ (\alpha)\in\mathbb{F}_{n_1}^+\times\cdots\times\mathbb{F}_{n_k}\right\}$.

We mention that, under the additional condition that $\overline{\text{span}}\{\mathbf{W}_{(\alpha)}\mathbf{W}_{(\beta)}^*:\ (\alpha),(\beta)\in\mathbb{F}_{n_1}^+\times\cdots\times\mathbb{F}_{n_k}^+\}$ is equel to $C^*(\mathbf{W}_{i,j})$, (eg. for the polyball) the map $\Psi_{\mathbf{q},\mathbf{T}}$ in the proof of Theorem 7.2 is unique and the dilation of $\mathbf{T}$ is minimal, i.e., $\tilde{\mathcal{K}}$ is the closed span of all $\mathbf{V}_{(\alpha)}\mathcal{H}$, $(\alpha)\in\mathbb{F}_{n_1}^+\times\cdots\times\mathbb{F}_{n_k}^+$. Taking into account the uniqueness of the minimal Stinespring representation and the Wold type decomposition mentioned above, one can prove the uniqueness, up to unitary equivalence, of the minimal dilation provided by Theorem 7.2. Moreover, let $\mathbf{T}'=(T_1',\ldots,T_k')$ be another k-tuple in $\mathbf{D_q^m}(\mathcal{H}')$ and let $\mathbf{V}'=(V_1',\ldots,V_k')$ be the corresponding dilation. Using standard arguments concerning the representation theory of C^*-algebras, one can prove that $\mathbf{T}$ and $\mathbf{T}'$ are unitarily equivalent if and only if $\dim\overline{\mathbf{\Delta_{q,T}^m}(I)(\mathcal{H})}=\dim\overline{\mathbf{\Delta_{q,T'}^m}(I)(\mathcal{H}')}$ and there are unitary operators $U:(\otimes_{i=1}^k F^2(H_{n_i}))\otimes\overline{\mathbf{\Delta_{q,T}^m}(I)(\mathcal{H})}\to(\otimes_{i=1}^k F^2(H_{n_i}))\otimes\overline{\mathbf{\Delta_{q,T'}^m}(I)(\mathcal{H}')}$ and $\Gamma:\mathcal{K}_\pi\to\mathcal{K}_{\pi'}$ such that

$$U(\mathbf{W}_{i,j}\otimes I_{\overline{\mathbf{\Delta_{q,T}^m}(I)(\mathcal{H})}})=(\mathbf{W}_{i,j}\otimes I_{\overline{\mathbf{\Delta_{q,T'}^m}(I)(\mathcal{H}')}})U,\qquad \Gamma\pi(\mathbf{W}_{i,j})=\pi'(\mathbf{W}_{i,j})\Gamma$$

for any $i\in\{1,\ldots,k\}$ and $j\in\{1,\ldots,n_i\}$, and $\begin{bmatrix}U&0\\0&\Gamma\end{bmatrix}\mathcal{H}=\mathcal{H}'$.

Corollary 7.4. *Let $\mathbf{V}:=(V_1,\ldots,V_k)\in\mathbf{D_q^m}(\tilde{\mathcal{K}})$ be the dilation of $\mathbf{T}:=(T_1,\ldots,T_k)\in\mathbf{D_q^m}(\mathcal{H})$, given by Theorem 7.2. Then,*

(i) $\mathbf{V}$ *is a pure element in* $\mathbf{D}_{\mathbf{q}}^{\mathbf{m}}(\tilde{\mathcal{K}})$ *if and only if* $\mathbf{T}$ *is a pure element in* $\mathbf{D}_{\mathbf{q}}^{\mathbf{m}}(\mathcal{H})$*;*

(ii) $(I-\Phi_{q_1,V_1})\cdots(I-\Phi_{q_k,V_k})(I_{\tilde{\mathcal{K}}})=0$ *if and only if* $(I-\Phi_{q_1,T_1})\cdots(I-\Phi_{q_k,T_k})(I_{\mathcal{H}})=0.$

Proof. According to Theorem 7.2, we have

$$(id-\Phi^{p_k}_{q_k,T_k})\cdots(id-\Phi^{p_1}_{q_1,T_1})(I_{\mathcal{H}})=P_{\mathcal{H}}\begin{bmatrix}(id-\Phi^{p_k}_{q_k,\mathbf{W}_k})\cdots(id-\Phi^{p_1}_{q_1,\mathbf{W}_1})(I_{\otimes_{i=1}^k F^2(H_{n_i})})\otimes I_{\overline{\mathbf{\Delta}^{\mathbf{m}}_{\mathbf{q},\mathbf{T}}(\mathcal{H})}} & 0\\ 0 & 0\end{bmatrix}|\mathcal{H}.$$

Hence, we deduce that $\lim_{\mathbf{q}=(q_1,\ldots,q_k)\in\mathbb{Z}_+^k}(id-\Phi^{p_k}_{q_k,T_k})\cdots(id-\Phi^{p_1}_{q_1,T_1})(I_{\mathcal{H}})=I$ if and only if $P_{\mathcal{H}}\begin{bmatrix}I&0\\0&0\end{bmatrix}|\mathcal{H}=I$. Consequently, $\mathbf{T}$ is pure if and only if $\mathcal{H}\perp(0\oplus\mathcal{K}_\pi)$. According to Theorem 7.2, this is equivalent to $\mathcal{H}\subset(\otimes_{i=1}^k F^2(H_{n_i}))\otimes\overline{\mathbf{\Delta}^{\mathbf{m}}_{\mathbf{q},\mathbf{T}}(I)(\mathcal{H})}$. On the other hand, since $(\otimes_{i=1}^k F^2(H_{n_i}))\otimes\overline{\mathbf{\Delta}^{\mathbf{m}}_{\mathbf{q},\mathbf{T}}(I)(\mathcal{H})}$ is reducing for each $V_{i,j}$, and $\widetilde{\mathcal{K}}$ is the smallest reducing subspace for $V_{i,j}$, which contains $\mathcal{H}$, we must have $\widetilde{\mathcal{K}}=(\otimes_{i=1}^k F^2(H_{n_i}))\otimes\overline{\mathbf{\Delta}^{\mathbf{m}}_{\mathbf{q},\mathbf{T}}(I)(\mathcal{H})}$. Therefore, item (i) holds.

To prove part (ii), note that

$$\mathbf{\Delta}^{\mathbf{m}}_{\mathbf{q},\mathbf{V}}(I_{\tilde{\mathcal{K}}})=\begin{bmatrix}\mathbf{\Delta}^{\mathbf{m}}_{\mathbf{q},\mathbf{W}}(I_{\otimes_{i=1}^k F^2(H_{n_i})})\otimes I_{\overline{\mathbf{\Delta}^{\mathbf{m}}_{\mathbf{q},\mathbf{T}}(\mathcal{H})}} & 0\\ 0 & 0\end{bmatrix}.$$

Hence, we deduce that $\mathbf{\Delta}^{\mathbf{m}}_{\mathbf{q},\mathbf{V}}(I_{\tilde{\mathcal{K}}})=0$ if and only if $\mathbf{\Delta}^{\mathbf{m}}_{\mathbf{q},\mathbf{W}}(I_{\otimes_{i=1}^k F^2(H_{n_i})})\otimes I_{\overline{\mathbf{\Delta}^{\mathbf{m}}_{\mathbf{q},\mathbf{T}}(\mathcal{H})}}=0$. On the other hand, we know that $\mathbf{\Delta}^{\mathbf{m}}_{\mathbf{q},\mathbf{W}}(I_{\otimes_{i=1}^k F^2(H_{n_i})})=\mathbf{P}_{\mathbb{C}}$. Consequently, the relation above holds if and only if $\mathbf{\Delta}^{\mathbf{m}}_{\mathbf{q},\mathbf{T}}=0$. Now, using Proposition 1.9, we obtain the equivalence in part (ii). The proof is complete. $\square$

We remark that every pure k-tuple $\mathbf{T}\in\mathbf{D}_{\mathbf{q}}^{\mathbf{m}}(\mathcal{H})$ with rank $\mathbf{\Delta}^{\mathbf{m}}_{\mathbf{q},\mathbf{T}}=1$ is unitarily equivalent to one obtained by compressing $(\mathbf{W}_1,\ldots,\mathbf{W}_n)$ to a co-invariant subspace under $\mathbf{W}_{i,j}$, where $i\in\{1,\ldots,k\}$ and $j\in\{1,\ldots,n_i\}$. Indeed, this follows from Theorem 7.2, Corollary 7.4, and the remarks preceding Corollary 7.4.

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